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以 LMM 利率模型評價利率衍生性商品：結合節點二項樹方法

Using the LIBOR Market Model to Price the Interest Rate Derivatives: A Recombining Binomial Tree Methodology

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摘要

LMM (LIBOR Market Model) 利率期限模型是一個很複雜的利率模型，要在此模型下推導評價封閉公式解或是數值評價方法都很困難。由於 LMM 模型具備非馬可夫的特性，建構利率樹時會發生節點無法重合的問題，而導致利率樹的節點個數呈爆炸性成長，使得電腦無法有效率地使用該利率樹進行評價。本篇論文利用 Ho, Stapleton, and Subrahmanyam (HSS) 介紹的造樹法來建構節點重合的LMM 利率樹。我們首先改寫 Poon 與 Stapleton (2005) 對 LMM 建構的離散時間數學模型，接下來推導遠期利率的條件期望值和條件變異數—這些資料對利率樹的建構十分重要。最後我們利用 HSS 提供的建樹法建構利率樹，並用該利率樹評價利率衍生性金融商品。第五節的數值資料驗證我們的數值模型可提供精確的評價結果。

【關鍵字】利率期限結構、LMM、結合節點二項樹

Abstract

LIBOR market model (LMM) is a complicated interest rate model and it is hard to be evaluated both analytically and numerically. Because of the non-Markov property of the LMM, a naively implemented tree model will not recombine. Thus the size of this naive tree model will grow explosively and the tree cannot be efficiently evaluated by computers. This paper proposes a recombining LMM tree model by taking advantages of tree construction methodology proposed by Ho, Stapleton, and Subrahmanyam (HSS). We first rewrite the discrete mathematical model for LMM suggested by Poon and Stapleton (2005). Then we derive the conditional means and the variances of the discrete forward rates which are important for the tree construction. Finally, our recombining trees for pricing interest rate derivatives are built by taking advantages of the tree construction methodology proposed by HSS. Numerical results illustrated in Section 5 suggest that our method can produce convergent and accurate pricing results for interest rate derivatives.

【Keywords】term structure, LMM, recombining tree

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1. Introduction

Many traditional interest rate models are based on instantaneous short rates and instantaneous forward rates. However, these rates can not be observed from the real world markets; consequently, it is hard to calibrate these models to fit the real world markets. LIBOR market model (LMM) is recently widely accepted in practice because it is based on the forward LIBOR rate which can be observed from the real world markets. This model was first proposed by Brace, Gatarek, and Musiela (1997) (abbreviate as BGM). In their model, the forward LIBOR rate is assumed to follow a lognormal distribution process, which makes the theoretical pricing formula for the caplet consists with the pricing formula under the Black's model (Black, 1976).

However, when implementing the LMM by a tree method, the tree will not recombine due to the non-Markov property of LMM. This non-recombining property makes the size of the tree grows explosively and thus the tree method is inefficient and difficult to price[1]. To address this problem, this paper adapts the HSS methodology proposed by Ho, Stapleton, and Subrahmanyam (1995) to construct a recombining binomial tree for LMM. By applying the HSS methodology into the LMM, the tree valuation method becomes feasible in pricing the interest rate derivatives.

The tree method proposed in this paper makes us have not to rely on the Monte Carlo simulation because our tree-based method is more accurate and efficient. Besides, the tree method can deal with American-style features, such as early exercise or early redemption, which is an intractable problem in Monte Carlo simulation.

The paper is structured as follows. Section 2 reviews important interest rate models. Section 3 introduces the market conventions about LMM and derives the drift of discrete-time version of LMM which follows the development in Poon and Stapleton (2005). In section 4, we introduce the HSS recombining node methodology (Ho et al., 1995) into the discrete-time version of LMM which derived in section 3 and construct the pricing model. The numerical pricing results and the sensitive analyses in section 5 verify the correctness and robustness of our tree model. Finally, section 6 concludes the paper.

2. Review of Interest Rate Models

In this section we introduce some important interest rate models that can be generally

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1 Similarly, HJM model also has the non-Markov property and thus the tree for HJM grows explosively as mentioned in Hull (2006).

categorized into two categories: equilibrium models and no-arbitrage models. Equilibrium models usually start with assumptions about economic variables and derive a stochastic process for the short rate *r*. On the other hand, a no-arbitrage model makes the behaviors of interest rate exactly consist with the initial term structure of interest rates. We will introduce some important equilibrium models first.

Vasicek model, suggested in Vasicek (1977), assumes that the short rate process *r* (*t*) follows the Ornstein-Uhlenbeck process and has the following expression under the risk-neutral measure:

$$dr(t) = \alpha(\beta - r(t))dt + \sigma dW(t),$$

where mean reversion rate α , average interest level β, and volatility σ are constants. Note that the short rate *r* (*t*) appears to be pulled back to long-run average interest level β, which is called mean reversion property. Vasicek (1977) shows that the price at time t of a zero-coupon bond that pays $1 at time *T* can be expressed as

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$B(t,T) = \frac{1-e^{-\alpha(T-t)}}{\alpha}$$

$$A(t,T) = \exp[\frac{(B(t,T)-T+t)(\alpha^2\beta-\sigma^2/2)}{\alpha^2} - \frac{\sigma^2 B(t,T)^2}{4\alpha}].$$

The drawback of Vasicek model is that the short rate could be negative. To improve this drawback, Cox, Ingersoll, and Ross (1985) propose CIR model which makes the short rate *r* (*t*) always non-negative. Under the risk neutral measure, *r* (*t*) follows the following process:

$$dr(t) = \alpha(\beta - r(t))dt + \sigma\sqrt{r(t)}dW(t),$$

which also has the mean reversion property. Moreover, to make the short rate nonnegative, CIR model use a non-constant volatility $\sigma\sqrt{r(t)}$ to replace the constant volatility in Vasicek model. The zero-coupon bond price *P* (*t*,*T*) in CIR model can be expressed as follows:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

where

$$B(t,T) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\alpha)(e^{\gamma(T-t)}-1)+2\gamma},$$

$$A(t,T)=[\frac{2\gamma e^{(\alpha+\gamma)(T-t)/2}}{(\gamma+\alpha)(e^{\gamma(T-t)}-1)+2\gamma}]^{2\alpha\beta/\sigma^2},$$

and $\gamma=\sqrt{\alpha^2+2\sigma^2}$.

Note that equilibrium models cannot exactly fit prevailing term structure of interest rates. Thus, no-arbitrage models are designed to calibrate prevailing term structure of interest rates. We first focus on instantaneous short rate models.

Ho and Lee (1986) propose the first no-arbitrage model. The short rate process of Ho-Lee model under the risk-neutral measure is as follows:

$$dr(t)=\theta(t)dt+\sigma dW(t),$$

where $\theta(t)$ is a function of time chosen to ensure that the model fits the initial term structure, and it can be expressed by the instantaneous forward rate as follows:

$$\theta(t)=f_t(0,t)+\sigma^2 t,$$

where f_t (0, t) is the instantaneous forward rate for maturity t as seen at time zero and the subscript t denotes a partial derivative with respect to t. Moreover, the price of the zero-coupon bond P (t,T) in Ho-Lee model can be expressed as

$$P(t,T)=A(t,T)e^{-r(t)(T-t)}$$

where

$$\ln A(t,T)=\ln\frac{P(0,T)}{P(0,t)}+(T-t)f(0,t)-\frac{1}{2}\sigma^2 t(T-t)^2.$$

Hull and White (1990) then provide a generalized version of the Vasicek model and it provides an exact fit to the prevailing term structure. The short rate process of the Hull-White model is

$$dr(t)=[\theta(t)-\alpha r(t)]dt+\sigma dW(t),$$

where α and σ are constants and the function of $\theta(t)$ can be calculated from the initial term structure as follows:

$$\theta(t)=f_t(0,t)+\alpha f(0,t)+\frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t})$$

The zero-coupon bond price P (t,T) in Hull-White model has the same general form as in

Vasicek model:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$B(t,T) = \frac{1-e^{-\alpha(T-t)}}{\alpha}$$

$$\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t)} + B(t,T)f(0,t) - \frac{1}{4\alpha^3}\sigma^2(e^{-\alpha T} - e^{-\alpha t})^2(e^{2\alpha t} - 1)$$
.

On the other hand, Heath, Jarrow, and Morton (1992) model the stochastic process of the instantaneous forward rate to describe the evolution of the entire yield curve in continuous time. The instantaneous forward rate $f\,(t,T)$ for the fixed maturity T under the risk-neutral measure is described as follows:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

where

$W(t) = (W_1(t),\cdots,W_d(t))$ is a d-dimensional Brownian motion,

$\sigma(t,T) = (\sigma_1(t,T),\cdots,\sigma_d(t,T))$ is a vector of adapted processes,

$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds = \sum_{i=1}^{d}\sigma_i(t,T)\int_t^T \sigma_i(t,s)ds$.

Given the dynamics of the instantaneous forward rate $f\,(t,T)$, the Ito's lemma can be applied to obtain the dynamics of the zero-coupon bond price $P\,(t,T)$:

$$dP(t,T) = P(t,T)[r(t)dt - (\int_t^T \sigma(t,s))dW(t)]$$
,

where $r\,(t)$ can be expressed as follows:

$$r(t) = f(t,t) = f(0,t) + \int_0^t \sigma(u,t)\int_u^t \sigma(u,s)dsdu + \int_0^t \sigma(s,t)dW(s)$$
.

Note that the short rate process $r\,(t)$ in the HJM model is non-Markov and makes a naively-implemented tree for simulating the short rate process a non-recombined tree. Besides, another drawback of the HJM model is that it is expressed in terms of instantaneous forward rates, which can not be directly observed in the market. Thus, it is difficult to calibrate the HJM model to price the actively traded instruments.

To address the aforementioned problem, Brace et al. (1997) suggest the BGM model that models the dynamics of the forward rates. However, Miltersen, Sandmann, and

Sondermann (1997) discover this model independently, and Jamshidian (1997) also contributes significantly to its initial development. To reflect the contribution of multiple authors, many practitioners, including Rebonato (2002), renamed this model to LIBOR market model (LMM).

There are two common versions of the LMM, one is the lognormal forward LIBOR model (LFM) for pricing caps and the other is the lognormal swap model (LSM) for pricing swaptions. The LFM assumes that the discrete forward LIBOR rate follows a lognormal distribution under its own numeraire, while the LSM assumes that the discrete forward swap rate follows a lognormal distribution under the swap numeraire. The two assumptions do not match theoretically, but lead to small discrepancies in calibrations using realistic parameterizations. The following derivations are based on the LFM.

Unlike HJM that models $f(t,T)$, the instantaneous forward rate at time T as seen at time t, the LFM models $f(t;T_i,T_{i+1})$, the discrete forward rate seen at time t for the period between time T_i and time T_{i+1}. $f(t;T_i,T_{i+1})$ follows a zero-drift stochastic process under its own forward measure:

$$\frac{df(t;T_i,T_{i+1})}{f(t;T_i,T_{i+1})}=\sigma_i(t)dW_i(t),$$

where $dW_i(t)$ is a Brownian motion under the forward measure $Q^{T_{i+1}}$ defined with respect to the numeraire asset $P(t,T_{i+1})$, and $\sigma_i(t)$ measures the volatility of the forward rate process. Using Ito's lemma, the stochastic process of the logarithm of the forward rate is given as follows:

$$d\ln f(t;T_i,T_{i+1})=\frac{-\sigma_i^2(t)}{2}dt+\sigma_i(t)dW_i(t) \tag{1}$$

The stochastic integral of equation (1) can be given as follows. For all $0\leq t\leq T_i$,

$$\ln f(t;T_i,T_{i+1})=\ln f(0;T_i,T_{i+1})-\int_0^t\frac{-\sigma_i^2(u)}{2}du+\int_0^t\sigma_i(u)dW_i(u) \tag{2}$$

Since the volatility function $\sigma_i(t)$ is deterministic, the logarithm of the forward rate is normally distributed, implying that the forward rate is lognormally distributed. For $t=T_i$, equation (2) implies that the future LIBOR rate $L(T_i,T_{i+1})=f(T_i;T_i,T_{i+1})$ is also lognormally distributed. This explains why this model is called the lognormal forward LIBOR model. Thought each forward rate is lognormally distributed under its own forward measure, it is not lognormally distributed under other forward measure.

3. Market Conventions of the LMM and the Discrete-Time Version of the LMM

In this section, we first introduce some market conventions of LMM and related interest rate instruments such as caplets and forward rate agreements (FRAs). Next, we restate some key results in the Poon and Stapleton (2005) and then re-derive generalized formulas for discrete-time version of the LMM that can be directly used in our tree construction procedure.

3. 1 Market Conventions of the LMM

The relationship between the discrete LIBOR rate $L\,(T_i\,,T_{i+1})$ for the term $\delta_i = T_{i+1} - T_i$ and the zero-coupon bond price $P\,(T_i\,,T_{i+1})$ is given as follows:

$$P(T_i, T_{i+1}) = \frac{1}{1+\delta_i L(T_i, T_{i+1})} \quad , \tag{3}$$

where $t \leq T_0 < T_1 < T_2 < \cdots < T_n$ is the time line and δ_i is called the tenor or accrual fraction for the period T_i to T_{i+1}.

The time t discrete forward rate for the term $\delta_i = T_{i+1} - T_i$ is related to the price ratio of two zero-coupon bonds maturing at times T_i and T_{i+1} as follows:

$$1+\delta_i f(t;T_i, T_{i+1}) = \frac{P(t,T_i)}{P(t,T_{i+1})} \quad . \tag{4}$$

The forward rate converges to the future LIBOR rate at time T_i , or:

$$\lim_{\tau \to T_i} f(\tau; T_i, T_{i+1}) = L(T_i, T_{i+1}) \quad .$$

We can rewrite equation (4) as follows:

$$f(t;T_i,T_{i+1})P(t,T_{i+1}) = \frac{1}{\delta_i}[P(t,T_i) - P(t,T_{i+1})] \, .$$

Then, we define basic terms that are frequently used in the market as follows:

$For\,(t,\,T_1\,,T_n)$: the forward price at time t to invest a zero coupon bond matured at time T_n at time T_1 and can be expressed as $P\,(t,\,T_n) \,/\, P\,(t,\,T_1)$.

$y\,(t,\,T_1)$: the annual yield rate at time t to T_1 time and its relation with the zero coupon bond is givens as $P\,(t,\,T_n) = 1 \,/\, (1+\,\delta_1 y\,(t,\,T_1))$.

$f\,(t;\,T_n\,,T_{n+1})$: the forward rate at time t for the time period T_n to T_{n+1} and its relation with forward price of a zero coupon bond is given as $For\,(t;T_n\,,T_{n+1}) = 1 \,/\, (1+\,\delta_n f\,(t;T_n,T_{n+1}))$.

After introducing these basic terms, we introduce a popular interest rate option- an interest rate cap. A cap is composed of a series of caplets. For a T_i-maturity caplet, the practitioners widely use the Black's formula to obtain its value at time t as follows:

$$caplet_i(t) = A \times \delta_i \times P(t, T_{i+1})[f(t; T_i, T_{i+1})N(d_1) - KN(d_2)], \tag{6}$$

where

$$d_1 = \frac{\ln(f(t; T_i, T_{i+1})/K) + \sigma_i^2 (T_i - t)/2}{\sigma_i \sqrt{T_i - t}},$$

$$d_2 = \frac{\ln(f(t; T_i, T_{i+1})/K) - \sigma_i^2 (T_i - t)/2}{\sigma_i \sqrt{T_i - t}},$$

A : the notional value of the caplet,

δ_i : the length of the interest rate reset interval as a proportion of a year,

$P(t, T_{i+1})$: the zero coupon bond price paying 1 unit at maturity date T_{i+1} ,

K : the caplet strike price,

σ_i : the Black implied volatility of the caplet,

$N(.)$: the cumulative probability distribution function for a standardized normal distribution.

Furthermore, under the LIBOR basis, we can derive the same theoretical pricing equation for the caplet as equation (6) from the LFM model. Because both of LFM and Black's model (Black, 1976) are assuming that the forward rate follows the lognormal distribution and we get the consistent results.

Another instrument we illustrate here as a key to derive out the discrete-time version of the LMM is the forward rate agreement (FRA). A FRA is an agreement made at time t to exchange fixed-rate interest payments at a rate K for variable rate payments, on a notional amount A, for the loan period T_n to T_{n+1} equal to one year. The settlement amount at time T_n on a long FRA is

$$FRA(T_n) = \frac{A(y(T_n, T_{n+1}) - K)}{1 + y(T_n, T_{n+1})}, \tag{7}$$

where $y(T_n, T_{n+1})$ is the annual yield at time T_n to T_{n+1} . At the time of the contract inception, a FRA is normally structured so that it has zero value. To avoid the arbitrage, the strike rate K is set equal to the market forward rate $f(t; T_n, T_{n+1})$. We denote the value of the FRA at time t as $FRA(t, T_n)$ which can be expressed as

$$FRA(t, T_n) = E_t\left[\frac{A(y(T_n, T_{n+1}) - f(t; T_n, T_{n+1}))}{1 + y(T_n, T_{n+1})}\right] = 0 . \tag{8}$$

3.2 The Discrete-Time Version of the LMM

We first restate the most important results which are under the "risk-neutral" measure in the Poon and Stapleton (2005).

(A) For a zero-coupon bond price is given by

$$P(t,T_n)=P(t,T_1)E_t(P(T_1,T_n)), \tag{9}$$

or we can write

$$E_t(P(T_1,T_n))=\frac{P(t,T_n)}{P(t,T_1)}=For(t,T_1,T_n).$$

(B) The drift of the forward bond price is given by

$$\begin{aligned}&E_t[For(T_1,T_i,T_n)]-For(t,T_i,T_n)\\&\quad=-\frac{P(t,T_1)}{P(t,T_n)}\text{cov}_t[For(T_1,T_i,T_n),P(T_1,T_n)].\end{aligned} \tag{10}$$

(C) The drift of T_n-period forward rate is obtained from the equation (8) and given by

$$\begin{aligned}&E_t[f(T_1;T_n,T_{n+1})]-f(t;T_n,T_{n+1})=\\&-\text{cov}_t[f(T_1;T_n,T_{n+1}),\frac{1}{1+y(T_1,T_2)}\times\frac{1}{1+f(T_1;T_2,T_3)}\times\cdots\times\frac{1}{1+f(T_1;T_n,T_{n+1})}]\\&\times(1+f(t;T_1,T_2))\cdot(1+f(t;T_2,T_3))\cdot\cdots\cdot(1+f(t;T_n,T_{n+1}))\end{aligned} \tag{11}$$

Now we re-derive the generalized version for above formulas so that the results can be directly used in our tree construction. We first apply the results to the LIBOR basis for the FRA and rewrite the equation (7) as follows

$$FRA(T_n)=\frac{A(f(T_n;T_n,T_{n+1})-K)\cdot\delta_n}{1+\delta_n f(T_n;T_n,T_{n+1})}, \tag{12}$$

where $\delta_n = T_{n+1} - T_n$. We further assume all the tenors are the same (i.e. $\delta_1 = \delta_2 = ... = \delta_n = \delta$) and the notional amount A equal to one to make the equation briefer. By using the above results and similar steps, we derive out the FRA value at time t of the equation (12) to generalize the T_n-maturity forward rate

$$
\begin{aligned}
&E_t[f(T_1;T_n,T_{n+1})]-f(t;T_n,T_{n+1})= \\
&\frac{-1}{\delta}\operatorname{cov}_t[\delta f(T_1;T_n,T_{n+1}),\frac{1}{1+\delta f(T_1;T_1,T_2)}\cdots\cdots\frac{1}{1+\delta f(T_1;T_n,T_{n+1})}] \\
&\times(1+\delta f(t;T_1,T_2))\cdot(1+\delta f(t;T_2,T_3))\cdots\cdots(1+\delta f(t;T_n,T_{n+1}))
\end{aligned} \tag{13}
$$

We assume that the forward rate $f(T_1;T_n,T_{n+1})$ is the lognormal for all forward maturities, T_n . Then, we use the approximate result for the covariance term, that is for the small change around the value $X = a$, $Y = b$, we have cov $(X, Y) \approx ab$ cov (ln X, ln Y). Here we take $a = f(t;T_1,T_2)$ and $b = 1/(1+f(t;T_1,T_2))$ to evaluate $\operatorname{cov}_t(f(T_1;T_1,T_2), \frac{1}{1+f(T_1;T_1,T_2)}$, then we have

$$
\begin{aligned}
&\operatorname{cov}_t(f(T_1;T_1,T_2),\frac{1}{1+f(T_1;T_1,T_2)})= \\
&f(t;T_1,T_2)(\frac{1}{1+f(t;T_1,T_2)})\operatorname{cov}_t(\ln y(T_1,T_2),\ln\frac{1}{1+y(T_1,T_2)}) .
\end{aligned}
$$

Substitute it into the equation (13) and use the property of logarithms to express the drift of T_n -maturity forward rate as the sum of a series of covariance terms. Finally, to make our covariance terms in a recognizable form, we use the extension of Stein's lemma to evaluate the term with a form $\operatorname{cov}_t(\ln f(T_1;T_n,T_{n+1}), \ln(\frac{1}{1+f(T_1;T_1,T_2)}))$.

Stein's Lemma for lognormal variables

For joint-normal variables x and y, we have

$$\operatorname{cov}(x,g(y))=E(g'(y))\cdot\operatorname{cov}(x,y) .$$

Hence, if $x = \ln X$ and $y = \ln Y$, then

$$\operatorname{cov}(\ln X,\ln\frac{1}{1+Y})=E(\frac{-Y}{1+Y})\cdot\operatorname{cov}(\ln X,\ln Y) .$$

Thus we have

$$
\begin{aligned}
&\operatorname{cov}_t(\ln f(T_1;T_n,T_{n+1}),\ln(\frac{1}{1+f(T_1;T_1,T_2)}))= \\
&E_t(\frac{-f(T_1;T_1,T_2)}{1+f(T_1;T_1,T_2)})\operatorname{cov}_t(\ln f(T_1;T_n,T_{n+1}),\ln f(T_1;T_1,T_2))
\end{aligned}
$$

Here, we apply the aforementioned result to the equation (13) and derive out the drift of the forward LIBOR rate as the sum of a series of covariance terms as follows:

$$\begin{aligned} &E_t[f(T_1;T_n,T_{n+1})]-f(t;T_n,T_{n+1})= \\ &\quad f(t;T_n,T_{n+1})\times\frac{\delta f(t;T_1,T_2)}{1+\delta f(t;T_1,T_2)}\cdot \mathrm{cov}_t[\ln f(T_1;T_n,T_{n+1}),\ln f(T_1;T_1,T_2)] \\ &\quad +\cdots \\ &\quad + f(t;T_n,T_{n+1})\times\frac{\delta f(t;T_n,T_{n+1})}{1+\delta f(t;T_n,T_{n+1})}\cdot \mathrm{cov}_t[\ln f(T_1;T_n,T_{n+1}),\ln f(T_1;T_n,T_{n+1})] \end{aligned} \tag{14}$$

We also assume that the covariance structure is inter-temporally stable and is a function of the forward maturities and $\mathrm{cov}_t\,[\ln f\,(T_1;T_i\,,T_{i+1}),\,\ln f\,(T_1;T_n\,,T_{n+1})]$ is not dependent on t. Then we define

$$\mathrm{cov}_t[\ln f(T_1;T_i,T_{i+1}),\ln f(T_1;T_n,T_{n+1})]\equiv\tilde{\sigma}_{i,n} \qquad i=1,\ 2,\cdots,n$$

where $\tilde{\sigma}_{i,n}$ is the covariance of the log i-period forward LIBOR and the log n-period forward LIBOR. Finally, we can rewrite equation (14) as follows:

$$\begin{aligned} \frac{E_t[f(T_1;T_n,T_{n+1})]-f(t;T_n,T_{n+1})}{f(t;T_n,T_{n+1})} &= \frac{\delta f(t;T_1,T_2)}{1+\delta f(t;T_1,T_2)}\cdot\tilde{\sigma}_{1,n}+\frac{\delta f(t;T_2,T_3)}{1+\delta f(t;T_2,T_3)}\cdot\tilde{\sigma}_{2,n} \\ &\quad +\cdots+\frac{\delta f(t;T_n,T_{n+1})}{1+\delta f(t;T_n,T_{n+1})}\cdot\tilde{\sigma}_{n,n} \end{aligned} \tag{15}$$

4. Introducing the HSS Recombining Node Methodology and Applying It to Construct a Recombining LMM Tree

Ho et al. (1995) suggest a general methodology for creating a recombining multi-variate binomial tree to approximate a multi-variate lognormal process. Our assumption about the LMM satisfies the required conditions of the HSS methodology. Therefore, we apply the HSS methodology to construct the recombining trees for LMM. We first introduce the HSS methodology and then apply it to construct a recombining LMM tree.

4.1 The HSS Methodology

The HSS method assumes the price of underlying asset X follows a lognormal diffusion process:

$$d \ln X(t) = \mu(X(t),t)dt + \sigma(t)dW(t), \tag{16}$$

where μ and σ are the instantaneous drift and volatility of $\ln X$, respectively, and *dw* (*t*) is a standard Brownian motion. They denote the unconditional mean at time 0 of the logarithmic asset return at time t_i as μ_i . The conditional volatility over the period t_{i-1} to t_i is denoted $\sigma_{i-1,i}$ and the unconditional volatility is $\sigma_{0,i,}$.

To approximate the underlying asset process in equation (16) with a binomial process at time $t_i, i = 1, \cdots, m$, given the means μ_i , conditional volatilities $\sigma_{i-1,i}$, and the unconditional volatilities $\sigma_{0,i}$, the HSS method requires that the conditional volatilities of the binomial process $\hat{\sigma}_{i-1,i}(n_i)$, where n_i denotes the number of binomial stages between time t_{i-1} and time t_i , converges to the conditional volatility ln *X* of as follows:

$$\lim_{n_i \to \infty} \hat{\sigma}_{i-1,i}(n_i) = \sigma_{i-1,i}, \quad \forall i. \tag{17}$$

Similarly, the unconditional volatility $\hat{\sigma}_{0,i}(n_1, n_2, \cdots, n_i)$ and the mean $\hat{\mu}_i$ of the binomial process converge to the unconditional volatility and the mean of ln *X* as follows:

$$\lim_{n_1, n_2 \ldots n_i \to \infty} \hat{\sigma}_{0,i}(n_1, n_2, \cdots, n_i) = \sigma_{0,i}, \quad \forall i. \tag{18}$$

$$\lim_{n_i \to \infty} \hat{\mu}_i = \mu_i. \tag{19}$$

The HSS method involves the construction of m separate binomial distributions for the prices of the underlying asset at time $t_1, \cdots, t_i, \cdots, t_m$, and has the set of a discrete randomness for X_i , where X_i is only defined at time t_i . In general they have the form of X_i at node *r*:

$$X_{i,r} = X_0 u_i^{N_i - r} d_i^r, \tag{20}$$

where $N_i = \sum_{l=1}^{i} n_l$. The upward and the downward movements u_i , d_i and the branching probabilities are properly selected to satisfy the equations (17), (18) and (19). They denote

$$x_i = \ln(X_i / X_0)$$

and the probability to reach x_i given a node $x_{i-1,r}$ at t_{i-1} as

$$q(x_i \mid x_{i-1} = x_{i-1,r}) \text{ or } q(x_i)$$

An example, where $m = 2$ and we have X_0, X_1 and X_2 is illustrated in Figure 1.

Lemma 1 Suppose that the up and down movements u_i and d_i are chosen so that

$$d_i = \frac{2(E(X_i)/X_0)^{\frac{1}{N_i}}}{1+\exp(2\sigma_{i-1,i}\sqrt{(t_i - t_{i-1})/n_i})}, \quad i = 1, 2, \cdots, m, \tag{21}$$

$$u_i = 2(E(X_i)/X_0)^{\frac{1}{N_i}} - d_i, \quad i = 1, 2, \cdots, m, \tag{22}$$

where $N_i = \sum_{l=1}^{i} n_l$, then if , for all i, the conditional probability $q\,(x_l) \to 0.5$ as $n_l \to \infty$, for $l = 1, \cdots, i$, then the unconditional mean and the conditional volatility of the approximated process approach respectively their true values:

$$\lim_{\substack{n_l \to \infty \\ l=1,\cdots,i}} \frac{\hat{E}(X_i)}{X_0} \to \frac{E(X_i)}{X_0}, \quad \lim_{n_i \to \infty} \hat{\sigma}_{i-1,i} \to \sigma_{i-1,i}$$

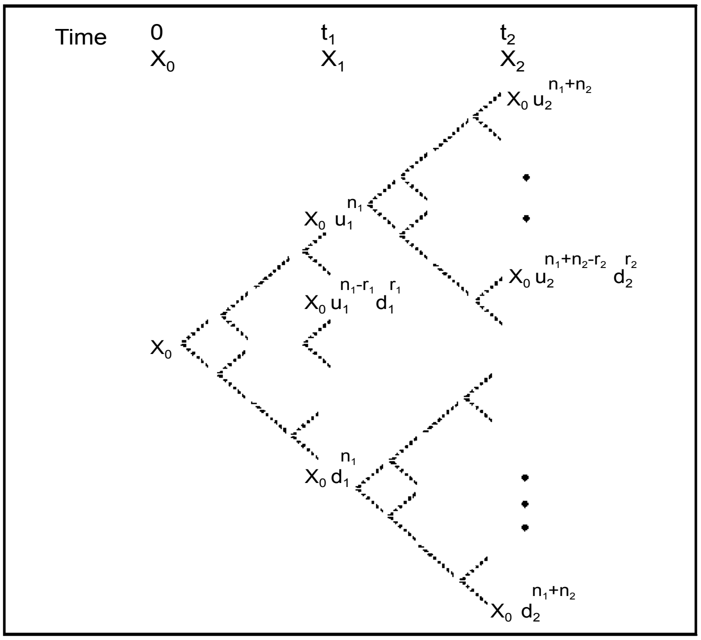

Figure 1 A discrete process for X_1, X_2

There are n_1+1 nodes at t_1 numbered r = 0, 1,..., n_1. There are n_1+ n_2+1 nodes at t_2 numbered r = 0, 1,..., n_1+ n_2. X_0 is the starting price, X_1 is the price at time t_1, and X_2 is the price at time t_2. u_1, d_1, u_2 and d_2 are the proportionate upward and downward movements.

Since x_i =ln (X_i / X_0) is normally distributed, it follows that the regression formula

$$x_i = a_i + b_i x_{i-1} + \varepsilon_i, \quad E_{i-1}(\varepsilon_i) = 0$$

is linear with

$$b_i = \sqrt{[t_i \sigma_{0,i}^2 - (t_i - t_{i-1})\sigma_{i-1,i}^2] / t_{i-1}\sigma_{0,i-1}^2},$$

and

$$a_i = E(x_i) - b_i E(x_{i-1})$$

They determined the conditional probabilities q (X_i) so that

$$E_{i-1}(x_i) = a_i + b_i x_{i-1,r}$$

held for the approximated variables x_i and x_{i-1} .

Theorem 1 Suppose that the X_i are joint lognormally distributed. If the X_i are approximated with binomial distributions with $N_i = N_{i-1} + n_i$ stages and u_i and d_i given by equations (21) and (22), and if the conditional probability of an upward movement at node r at time $i-1$ is

$$q(x_i \mid x_{i-1} = x_{i-1,r}) = \frac{a_i + b_i x_{i-1,r} - (N_{i-1} - r)\ln u_i - r \ln d_i}{n_i(\ln u_i - \ln d_i)} - \frac{\ln d_i}{\ln u_i - \ln d_i}, \quad \forall i,\ r \qquad (23)$$

then $\hat{\mu}_i \to \mu_i$ and $\hat{\sigma}_{0,i} \to \sigma_{0,i}$ and $\hat{\sigma}_{i-1,i} \to \sigma_{i-1,i}$ as $n_i \to \infty,\ \forall i$

4.2 Applying the HSS Methodology to the LMM

After introducing the HSS methodology, we now apply this methodology to construct recombining trees for LMM and make some change to satisfy our conventions. We have the following propositions.

Proposition 1 For the forward LIBOR rate which follows the lognormal distribution, we can choose the proper upward and downward movements to determine the i-th period of the T_n-maturity forward LIBOR rate and have the form

$$f(i;T_n,T_{n+1})_r = f(0;T_n,T_{n+1}) u_i^{N_i - r} d_i^r,\ i = T_1, T_2, \cdots, T_n \qquad (24)$$

where

$$d_i = \frac{2[E(f(i;T_n,T_{n+1})) / f(0;T_n,T_{n+1})]^{\frac{1}{N_i}}}{1 + \exp(2\sigma_{i-1,i}\sqrt{(T_i - T_{i-1})/n_i})} \qquad (25)$$

$$u_i = 2[E(f(i;T_n,T_{n+1}))/f(0;T_n,T_{n+1})] - d_i \tag{26}$$

$$N_i = N_{i-1} + n_i \tag{27}$$

r: node's number from top to bottom at time T_i

The structure of the binomial tree can be shown as Figure 2, with n_1 +1 nodes at T_1 numbered as r = 0,1,...,n_1 . There are n_1 +n_2 +1 nodes at T_2 numbered as r = 0,1,...,n_1 + n_2 . Here we write the forward rate f (0; 2,3) in abbreviated form f (0; 2) and take $n_1 = n_2 = 2$, $r_1 = r_2 = 2$.

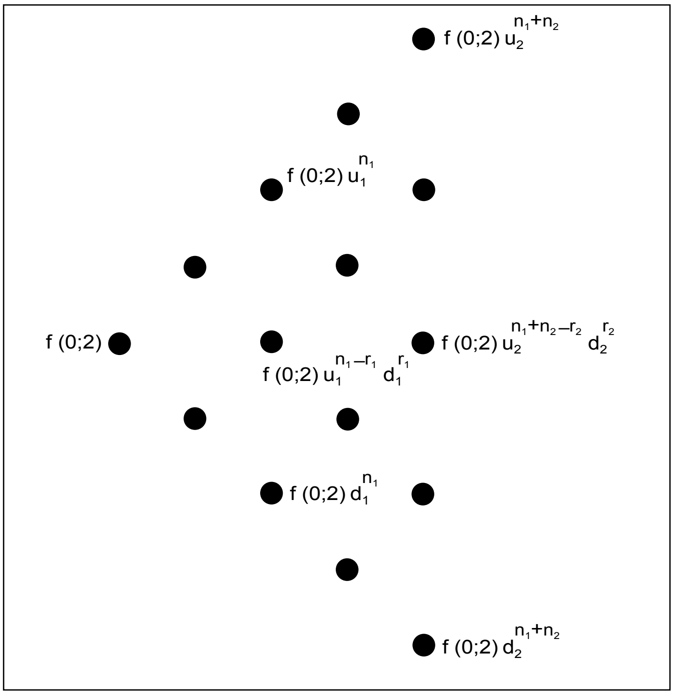

Figure 2 The binomial tree for the forward rate *f* (0;2,3)

After determining the structure of the forward LIBOR tree, we then have to choose the probability to satisfy the Proposition 1.

Proposition 2 Suppose that the forward LIBOR rate $f(i;T_n,T_{n+1})$ is joint lognormally distributed. If the forward rate $f(i;T_n,T_{n+1})$, $i = T_1,T_2,\cdots,T_n$ are approximated with binomial distributions with $N_i = N_{i-1} + n_i$ stages and u_i and d_i are given by equations (25) and (26), then the conditional probability of an upward movement at node r at time T_i is

$$q(x_i \mid x_{i-1} = x_{i-1,r}) = \frac{E_{i-1}(x_i) - (N_{i-1} - r)\ln u_i - r\ln d_i}{n_i(\ln u_i - \ln d_i)} - \frac{\ln d_i}{\ln u_i - \ln d_i} \quad \forall\ i,\ r, \tag{28}$$

where

$$x_i = \ln \frac{f(i;T_n,T_{n+1})}{f(0;T_n,T_{n+1})} \tag{29}$$

$$E_{i-1}(x_i) = a_i + b_i x_{i-1,r} = E(x_i) - b_i E(x_{i-1}) + b_i x_{i-1,r}\ . \tag{30}$$

To determine the conditional probability, we must derive $E_{i-1}(x_i)$ first. We first derive $E(x_i)$ term in equation (30). Since the forward rate $f(i;T_n,T_{n+1})$ is lognormally distributed, we have

$$E(x_i) = \ln\left[\frac{E(f(i;T_n,T_{n+1}))}{f(0;T_n,T_{n+1})}\right] - \frac{1}{2}\sigma_{0,i}^2 \tag{31}$$

Second, we use the result of equation (15) obtained from the last section, and rewrite it as follows:

$$\frac{E_t[f(T_1;T_n,T_{n+1})]}{f(t;T_n,T_{n+1})} = 1 + \frac{\delta_1 f(t;T_1,T_2)}{1+\delta_1 f(t;T_1,T_2)}\cdot\tilde{\sigma}_{1,n} + \frac{\delta_2 f(t;T_2,T_3)}{1+\delta_2 f(t;T_2,T_3)}\cdot\tilde{\sigma}_{2,n} + \cdots + \frac{\delta_n f(t;T_n,T_{n+1})}{1+\delta_n f(t;T_n,T_{n+1})}\cdot\tilde{\sigma}_{n,n} \tag{32}$$

Then multiple the $f(t;T_n,T_{n+1}) / f(0;T_n,T_{n+1})$ term on both side to get the general form of $E_t(f(T_1;T_n,T_{n+1})) / f(0;T_n,T_{n+1})$:

$$\frac{f(t;T_n,T_{n+1})}{f(0;T_n,T_{n+1})}\times\frac{E_t[f(T_1;T_n,T_{n+1})]}{f(t;T_n,T_{n+1})} = \frac{f(t;T_n,T_{n+1})}{f(0;T_n,T_{n+1})}\times\left(1+\frac{\delta_1 f(t;T_1,T_2)}{1+\delta_1 f(t;T_1,T_2)}\cdot\tilde{\sigma}_{1,n} + \frac{\delta_2 f(t;T_2,T_3)}{1+\delta_2 f(t;T_2,T_3)}\cdot\tilde{\sigma}_{2,n} + \cdots + \frac{\delta_n f(t;T_n,T_{n+1})}{1+\delta_n f(t;T_n,T_{n+1})}\cdot\tilde{\sigma}_{n,n}\right) \tag{33}$$

Finally, we substitute it into the formula (31) to obtain $E(x_i)$ term. Then, we take the value of $E(x_i)$ into equation (30) to compute the upward movement probability at time T_i given the node $f(i-1;T_n,T_{n+1})_r$.

Note that when n_l stages approach the infinite for $l = 1,\cdots,i$, the sum of n_l stages also approach the infinite (i.e.$N_i = \sum_{l=1}^{i} n_l \to \infty$) . The upward and downward movements and the conditional probability can be reduced to a briefer form which is easier to calculate. That is,

$$d_i = \frac{2}{1+\exp(2\sigma_{i-1,i}\sqrt{(T_i - T_{i-1})/n_i})} \quad ,$$

$$u_i = 2 - d_i \,,$$

and the conditional probability $q(x_l) \to 0.5$ as $n_l \to \infty$, for $l = 1, \cdots, i$.

5. Pricing Interest Rate Derivatives with LMM Recombining Trees

After constructing our recombining tree model for LMM by following the procedures mentioned in the last section, now we will use our tree model to price bond options and caplets and compare the pricing results with the Monte Carlo simulations. Some sensitive analyses are given in this section to verify the correctness and robustness of our model.

5.1 The Valuation of Bond Options on Zero Coupon Bonds in LMM

The bond option on a zero coupon bond (ZCB) is a bond that can be callable before maturity date with a callable price K. For example, we have a three years maturity zero coupon bond with a callable value K equal to 0.952381 dollar at year two. That is to say we can redeem the ZCB at year two with 0.952381 dollar or hold it until maturity at year three with 1 dollar. Therefore, we have to price the option value C_o of this callable bond at time 0 (see the Figure 3).

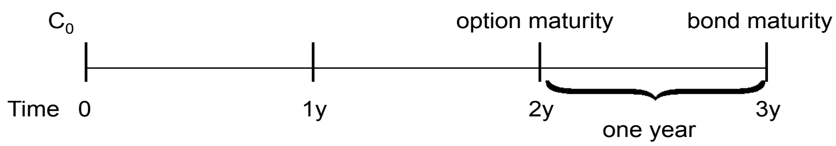

Figure 3 The callable bond for the 3-year maturity ZCB

To obtain the callable bond option value, we use the tree method to price the option value of the callable bond. The payoff function of the option is max $(P(2,3)-K,0)$, where $P(2,3)$ denotes the price of ZCB (that matures at year three) at year two. Then we discount the payoff function back to time 0 to get the option value of the callable bond. By assuming that the flat forward rate 5% and constant volatility 10%, the analytical value is

$$C_o = P(0, 2) \times E\,[\max(P(2, 3) - K, 0)]$$
$$= 0.90702948 \times 0.00258128$$

= 0.00234130,

which is close to 0.002342263685, the value generated by our recombining tree model. We further compare the relationship between the option value (denoted by the y-axis) and the volatility (denoted by the x-axis) in Figure 4.

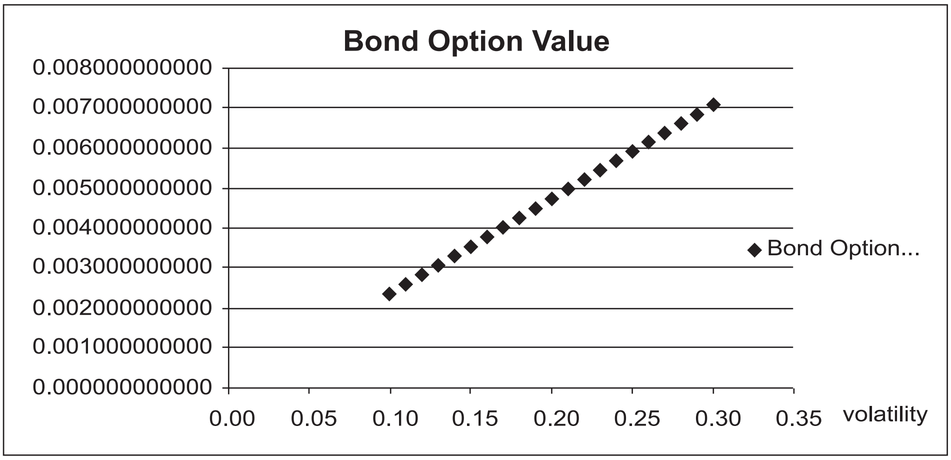

Figure 4 Bond option values for different volatility

From the above figure, we find that when the volatility increases, the value of bond option on ZCB increases. It is consistent with the inference for the Greek letter vega when the underlying asset's volatility increases the option value increases, too.

5.2 The Valuation of Bond Options on Coupon Bonds in LMM

Now we extend our tree model to price a European bond option on a coupon bond. The maturity and the strike price are 0.5 years and $100, respectively. The face value, the coupon rate, and the maturity of the bond are $100, 5%, and 0.75 years, respectively. The coupons are paid semiannually. We further assume the flat forward rate 5% and constant volatility 20%. The pricing results of our tree model are illustrated in Table 1. We use the Monte Carlo simulation illustrated in Table 2 as the benchmark. It can be observed that the pricing results of our tree model converge smoothly and accurately to 7.2564, the benchmark value generated by the Monte Carlo simulations with 1000000 trials. But it takes much lesser computational time (0.003 seconds) for the tree model than the Monte Carlo simulation (69 seconds).

Table 1 Pricing a bond option with binomial tree

n	tree	t (seconds)
10	7.256405610765942	0.000257
100	7.256407987405482	0.000620
1000	7.256408225544007	0.003150

"n" denotes the number of stages in our tree model, "tree" denotes the prices generated by our tree model. "t" measures the computational time in seconds.

Table 2 Pricing a bond option with the monte carlo simulation

m	MC	std	95% C.I.	t (seconds)
100	7.2622	0.030202	(7.203,7.3214)	0.012615
10000	7.2514	0.0035163	(7.2445,7.2583)	0.689756
1000000	7.2564	0.00034602	(7.2557,7.257)	69.853408

"m" denotes the number of trials for the Monte Carol Simulation, "MC" denotes the option prices generated by the Monte Carlo simulations, "std" denotes the standard error, "95% C.I." denotes the 95% confidence interval, and "t" measures the computational time in seconds.

To verify the robustness of our tree model, some sensitivity analyses are illustrated in Table 3 and Figure 5. Table 3 illustrates how the option values change when the forward rate curve shifts gradually from 0.05 to 0.055. Obviously, the option values decrease smoothly as the forward rates increase. The pricing results of our tree model also fall into the 95% confidence interval generated by the Monte Carlo simulation as illustrated in Figure 5.

Table 3 Sensitivity analysis: The impact of change of forward rate

r	tree	MC	std	95% C.I.
0.05	7.2564	7.2564	0.003439	(7.2497,7.2631)
0.0505	7.2303	7.2302	0.003471	(7.2234,7.2371)
0.051	7.2041	7.2041	0.003503	(7.1973,7.211)
0.0515	7.178	7.178	0.003535	(7.1711,7.185)
0.052	7.1519	7.152	0.003566	(7.145,7.159)
0.0525	7.1259	7.1259	0.003598	(7.1189,7.133)
0.053	7.0998	7.0999	0.003629	(7.0928,7.107)
0.0535	7.0738	7.0739	0.003661	(7.0667,7.0811)
0.054	7.0478	7.0479	0.003692	(7.0407,7.0552)
0.0545	7.0219	7.022	0.003724	(7.0147,7.0293)
0.055	6.9959	6.9961	0.003755	(6.9887,7.0034)

"r" denotes the forward rate, "tree" denotes the prices generated by our tree model with 1000 number of stages, "MC" denotes the option prices generated by the Monte Carlo simulations with 10000 trials, "std" denotes the standard error, and "95% C.I." denotes the 95% confidence interval. The numerical settings for the bond options are the same as the settings mentioned the first paragraph of Section 5.2.

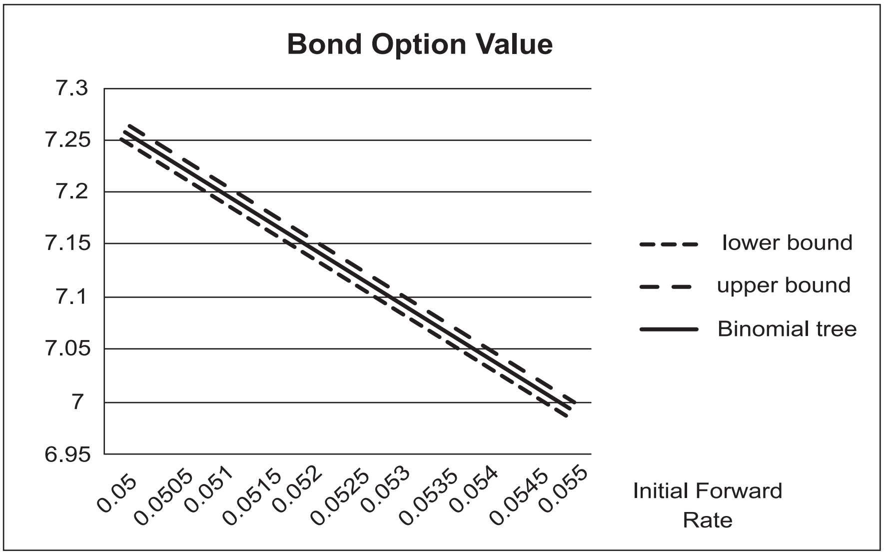

Figure 5 The change of bond value with respect to the change of initial forward rate

The x-axis and the y-axis denote the initial forward rate and the option value, respectively. The lower and the upper bounds denote the bounds of 95% confidence interval generated by the Monte Carlo simulation.

5.3 The Valuation of Caplets in LMM

A popular fixed income security is an interest rate cap, a contract that pays the difference between a variable interest rate applied to a principal and a fixed interest rate (strike price) applied to the same principal whenever the variable interest rate exceeds the fixed rate. We consider a cap with total life of T and let the tenor δ, the notional value A and the strike price K be fixed positive values. Note that the reset dates are T_1, T_2, ... , T_n and define $T_{n+1} = T$. Define the forward rate $f\,(T_i;T_i,T_{i+1})$ as the future spot interest rate for the period between T_i and T_{i+1} observed at time T_i $(1 \le i \le n)$. The payoff function of a caplet at time T_{i+1} is

$$A \times \delta \times \max(f(T_i;T_i,T_{i+1}) - K, 0) \tag{34}$$

Equation (34) is a caplet on the spot rate observed at time T_i with payoff occurring at time T_{i+1} . The cap is a portfolio consisted of n such call options (caplets).

To derive the price of the cap, we have to price the caplet first and then sum up the n caplets' values to get the price of a cap. For a caplet price at time *t*, we use the Black's formula mentioned in Section 3 (Equation (6)) to get the theoretical value as follows:

$$caplet_i(t) = A \times \delta_i \times P(t, T_{i+1})[f(t; T_i, T_{i+1})N(d_1) - KN(d_2)], \tag{35}$$

where

$$d_1 = \frac{\ln(f(t; T_i, T_{i+1})/K) + \sigma_i^2 (T_i - t)/2}{\sigma_i \sqrt{T_i - t}},$$
$$d_2 = \frac{\ln(f(t; T_i, T_{i+1})/K) - \sigma_i^2 (T_i - t)/2}{\sigma_i \sqrt{T_i - t}}.$$

After having the theoretical value as our benchmark, we use the payoff function to compute the price with our recombining tree method. To get the payoff function at time T_{i+1}, we have to know the evolution of the forward rate $f(0;T_i,T_{i+1})$ at time T_i. We construct the binomial tree of $f(0;T_i,T_{i+1})$ and known the $(f(T_i;T_i,T_{i+1})_r - K)^+$, r = 0, 1, ..., i. We first calculate the expectation of the payoff at time, and then multiply it with the value of ZCB ($P(t,T_{i+1})$) to get the caplet value at time *t*.

In the followings, we price a 10-period cap by computing each individual caplet matures from period 1 to period 10. We assume that the tenor δ and notional value *A* are equal to one and the volatility is equal to 10%. Here the strike price *K* is 5%, the forward curve is flat 5% and the stages n_i for every period are equal to 25. We calculate the value of one period caplet at time 0 ($caplet_1(0)$).

$$\begin{aligned} caplet_1(0) &= A \times \delta \times P(0,2) \times E[\max(f(1;1,2) - K, 0)] \\ &= 1 \cdot 1 \cdot P(0,2) \cdot 0.0020112666 \\ &= 0.90702948 \cdot 0.0020112666 \\ &= 0.0018242781 \end{aligned}$$

Table 4 Pricing caplets with volatility=10% and stage n_i = 25

Maturity	Black	Tree	Difference	Relative Difference (%)
1	0.0018085085	0.0018242781	0.0000157696	0.8719669656
2	0.0024348117	0.0024407546	0.0000059429	0.2440786919
3	0.0028388399	0.0028374958	-0.0000013441	0.0473484800
4	0.0031206153	0.0031282191	0.0000076038	0.2436631792
5	0.0033214311	0.0033204098	-0.0000010214	0.0307505629
6	0.0034637453	0.0034664184	0.0000026731	0.0771737036
7	0.0035616356	0.0035658574	0.0000042219	0.1185369137
8	0.0036247299	0.0036200240	-0.0000047059	0.1298286011
9	0.0036600091	0.0036633313	0.0000033221	0.0907678678
10	0.0036727489	0.0036743568	0.0000016079	0.0437804213
		RMSE	0.0000063671	

We assume that the tenor=1, the number of stages for every period is 25, volatility is 10%, and the forward curve is flat 5%.

Table 4 illustrates the pricing results for different maturity caplets. Besides the relative difference, we also use the RMSE to measure the difference between prices generated by our tree model and by the Black's model (Black, 1976), where RMSE is defined as follows:

RMSE (Root Mean Square Error)

A frequently-used measure of the differences between values predicted by a model or an estimator and the values actually observed from the thing being modeled or estimated. For the comparing difference between two models, the formula of RMSE can be expressed as

$$RMSE(\theta_1,\theta_2)=\sqrt{MSE(\theta_1,\theta_2)}=\sqrt{E((\theta_1-\theta_2)^2)}=\sqrt{\frac{\sum_{i=1}^{n}(x_{1,i}-x_{2,i})^2}{n}},$$

where

$$\theta_1=\begin{bmatrix}x_{1,1}\\x_{1,2}\\\vdots\\x_{1,n}\end{bmatrix} \quad\text{and}\quad \theta_2=\begin{bmatrix}x_{2,1}\\x_{2,2}\\\vdots\\x_{2,n}\end{bmatrix}$$

Here, θ_1 and θ_2 represent the prices generated by our tree model and the Black's model (Black, 1976), respectively.

Now we change the stages from 25 to 50 to figure out the relationship between stages and RMSE. The results are shown in Table 5.

Table 5 Pricing caplets with volatility=10% and stage n_i =50

Maturity	Black	Tree	Difference	Relative Difference (%)
1	0.0018085085	0.0018099405	0.0000014320	0.0791823392
2	0.0024348117	0.0024397802	0.0000049685	0.2040608652
3	0.0028388399	0.0028434461	0.0000046061	0.1622542164
4	0.0031206153	0.0031230795	0.0000024643	0.0789673074
5	0.0033214311	0.0033207434	-0.0000006878	0.0207066243
6	0.0034637453	0.0034620815	-0.0000016638	0.0480341136
7	0.0035616356	0.0035626664	0.0000010308	0.0289416315
8	0.0036247299	0.0036267433	0.0000020134	0.0555455781
9	0.0036600091	0.0036617883	0.0000017792	0.0486107386
10	0.0036727489	0.0036734197	0.0000006708	0.0182649876
		RMSE	0.0000025690	

We assume that the tenor=1, the number of stages for every period is 50, volatility is 10%, and the forward curve is flat 5%.

We also plot the RMSE with different stages between periods from 25 to 50 to see the convergence behavior of RMSE. Figure 6 shows that the convergence behavior of RMSE for the different stages. We find that when we increase stages between periods, both relative difference and RMSE decrease and RMSE converge to zero with the stages go to infinite.

To see the impact of volatility on the value of different caplets and the convergence behavior of RMSE, we change the volatility from 10% to 20%. We do the same procedures as we do in volatility 10%, and results for the 25 and 50 stages are presented in Table 6 and Table 7 respectively. Finally, we plot the RMSE with different stages from 25 to 50 for volatility 20% in Figure 6.

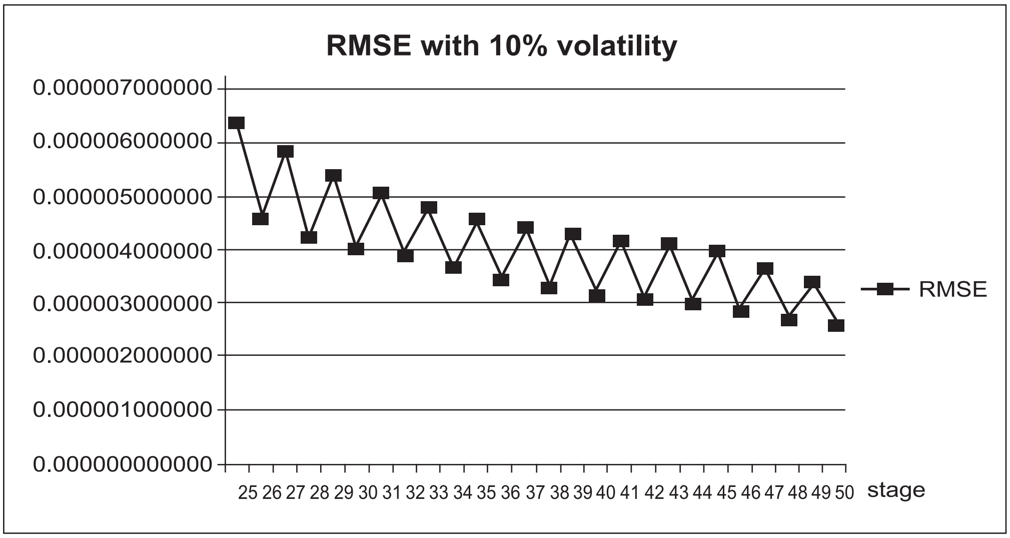

Figure 6 RMSE with volatility 10%

Table 6 Pricing caplets with volatility=20% and stage n_i =25

Maturity	Black	Tree	Difference	Relative Difference (%)
1	0.003612502	0.003629702	0.0000171997	0.4761162723
2	0.004857485	0.004880545	0.0000230598	0.4747275803
3	0.005656481	0.005664503	0.0000080217	0.1418144294
4	0.006210206	0.006193504	-0.0000167019	0.2689425754
5	0.006601645	0.006606642	0.0000049969	0.0756919625
6	0.006875986	0.006885656	0.0000096703	0.1406382089
7	0.007061574	0.007064876	0.0000033018	0.0467579666
8	0.007177804	0.007167478	-0.0000103258	0.1438570619
9	0.007238738	0.007241022	0.0000022831	0.0315395992
10	0.007255004	0.007260387	0.0000053836	0.0742052974
		RMSE	0.0000120045	

We assume that the tenor=1, the number of stages for every period is 25, volatility is 20%, and the forward curve is flat 5%.

Table 7 Pricing caplets with volatility=20% and stage n_i =50

Maturity	Black	Tree	Difference	Relative Difference (%)
1	0.0036125022	0.0036270258	0.0000145236	0.4020371797
2	0.0048574848	0.0048648548	0.0000073700	0.1517253656
3	0.0056564814	0.0056507415	-0.0000057399	0.1014746318
4	0.0062102056	0.0062167021	0.0000064965	0.1046097077
5	0.0066016455	0.0066036633	0.0000020178	0.0305658301
6	0.0068759861	0.0068743634	-0.0000016227	0.0235991644
7	0.0070615740	0.0070656730	0.0000040990	0.0580467885
8	0.0071778037	0.0071775957	-0.0000002080	0.0028984108
9	0.0072387385	0.0072386556	-0.0000000829	0.0011446687
10	0.0072550037	0.0072576598	0.0000026562	0.0366117075
		RMSE	0.0000060911	

We assume that the tenor=1, the number of stages for every period is 50, volatility is 20%, and the forward curve is flat 5%.

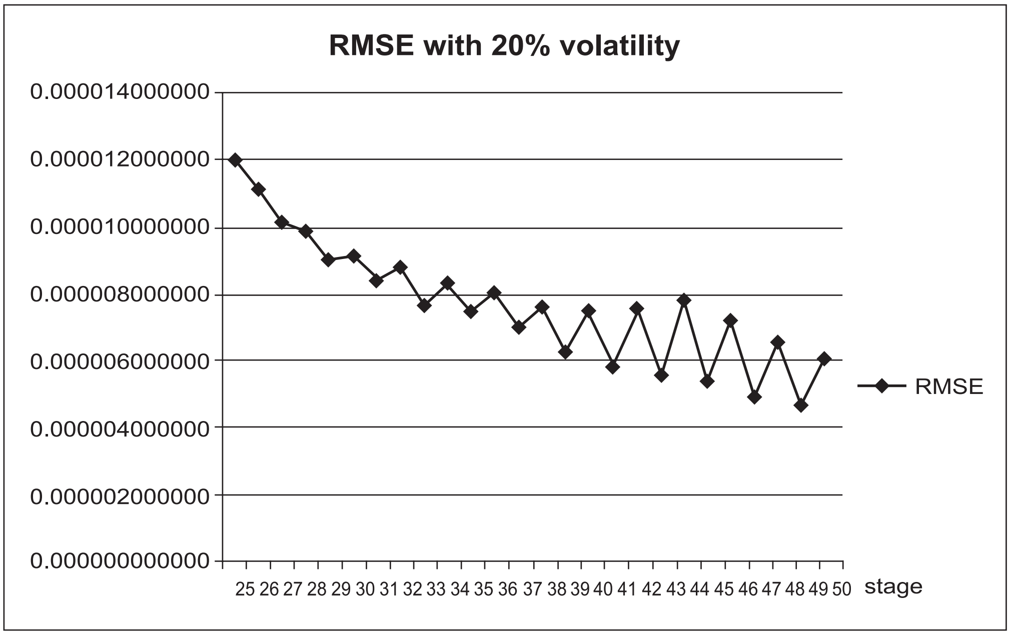

Figure 7 RMSE with volatility 20%

We find that when the volatility increases, the value of caplets increases, and the convergence rate of RMSE decreases. It is consistent with the observations in Lyuu (2002) that high volatility makes the option more valuable and makes the convergence rate slower.

6. Conclusions

Implementing the LIBOR market model with the tree method is difficult. To make the pricing procedure efficient, we construct a recombining-binomial-tree model to depict the evolution of the forward LIBOR rates. In our model, we have all the forward rates for the different maturity at any node of the recombining binomial tree. With these rates on the nodes, we can easily figure out the early exercise decision for the American-style derivatives which is a tough work in the Monte Carlo simulation.

After constructing the recombining binomial trees, the payoff of the interest rate derivatives on each node of the tree can be obtained. The values of the derivatives can be calculated by the backward induction method. We use the proposed model to calculate the values of bond options and caplets. The values generated by our tree model are very close to the theoretical values and the differences (or the pricing errors) decrease as the number of stages in our tree model increases.

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