A CONNECTED M-TREE RELAXATION FOR M-TRAVELLING SALESMEN PROBLEMS

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ABSTRACT

Utilizing branch-and-bound and relaxation techniques to solve large scale m-travelling salesmen problems to optimum needs strong lower bound procedures. Since the landmark 1-tree relaxation model devised by Held and Karp for travelling salesman problems, some relaxation models, such as: m-trees, augmented degree-constrained spanning trees have been developed for the solution of m-travelling salesman problems. This paper presents a new graphical structure, denoted as connected m-tree, to be a more promising relaxation model for m-travelling salesmen problems. Model, algorithm, and computational results are reported.

Key Words: Graph, Travelling Salesman, Algorithm, Relaxation

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I. INTRODUCTION

Because of the imbedded geographical directives in routing patterns' setting, routing problems, one of the combinatorial problems with high real-world practical value, but also with high intractability, have more structure than incorporated in their currnt models. In the solution of many combinatorial problems, the exploration of appropriate underlying graphical constructs often provides some advantages in their algorithm design, especially for the routing type problems [Ali & Huang 1988]. Held and Karp introduced the 1-tree structure for the travelling salesman problems (TSP), and took it as a relaxation for the model and solution design of TSP [Held & Karp 1970]. This landmarked discovery encouraged researchers to seek for better structures for those problems with similar structures: an m-tree structure is introduced by Ali & Kennington [Ali & Kennington 1986] in their duality-based branch-and-bound algorithm design to solve asymmetric multiple travelling salesmen problems (m TSP); augmented degree-consrained minimal spanning tree (DCMST) constructs have been employed by Gavish & Srikanth [Gavish & Srikanth 1986] as a relaxation for mTSP allowing immediate tours.

That the stronger the lower bound is, the more efficient the branch-and-bound procedure performs is a common rule for the utilization of relaxation technique in algorithm design of combinatorial type problems. This paper introduces a new subgraph denoted as connected m-tree to be a better relaxation model for mTSP, especially if the construction of solution does not allow the existence of immediate tours. In order to put the development in proper context. Section 2 contains a brief discussion of mTSP model. Section 3 presents various underlying subgraphs for mTSP as well as the new representation of mTSP in terms of those constructs. Section 4 then introduces the connected m-tree structure, together with the algorithm for its construction; Computational results and conclusions are then included in Section 5.

II. MODELS OF MULTIPLE TRAVELLING SALESMEN PROBLEMS

The multiple travelling salesmen problem is defined on a graph G[N,E] represented by the node set $N = \{0, 1, 2, \cdots, n\}$ and an arc set $E = \{e_{ij} \mid i, j \in N\}$. The set $N' = \{1, 2, \cdots, n\}$ with cardinality $\mid N' \mid = n$ is the set of non-base nodes and the node 0 is the base node. The arcs in $E = \{e_{ij} \mid i = 0 or j = 0\}$ are defined to be base arcs, and $E' = \{e_{ij} \mid i, j \in N'\}$ is the set of non-base arcs. Associated with each arc $e_{ij} \in E$, is a cost coefficient c_{ij} and a variable x_{ij} which has a value of 1 if the arc is used in a solution and 0 otherwise.

The mathematical representation of a symmetric multiple travelling salesmen problem defined on G[N,E] with cost vector $c = (c_{ij})_{i,j \in N}$ is specified as follows:

$$mTSP-1 \quad \min \sum_{i=1}^{i=n} \left(c_{0i} x_{0i} + \sum_{j=1, j \neq i}^{j=n} c_{ij} x_{ij} \right) \tag{2.1}$$

$$\text{s.t.} \sum_{j=1}^{j=n} x_{0j} = m \tag{2.2}$$

$$\sum_{j=1}^{j=n} x_{0j} = m \tag{2.3}$$

$$\sum_{j=0, j \neq i}^{j=n} x_{ij} + \sum_{j=0, j \neq i}^{j=n} x_{ij} = 2 \qquad for i = 1, 2, \cdots, n \tag{2.4}$$

$$\sum_{i,j \in S_k} x_{ij} = \mid S_k \mid -1 \qquad for\ any\ S_k \subset N', S_k \neq \emptyset \tag{2.5}$$

$$\sum_{i,j\in N'} x_{ij} = \mid N' \mid - m \tag{2.6}$$

$$x_{ij} = 0 \ or \ 1 \qquad for \ i,j \in N \tag{2.7}$$

Constraints (2.2) and (2.3) ensure that exactly $2m$ arcs are incident to the base node 0; (2.4) are the node-degree constraints specifying that each non-base node have exactly two arcs incident to it; Constraints (2.5), (2.6) and (2.7) define an m-component spanning forest. Thus feasible solutions are defined by: (i) subgraphs of: G[N,E], which have exactly $2m$ base arcs, and (n-m) non-base arcs which form an m-component spanning forest and; (ii) the node-degree constraints for tour formation. We denote the set of subgraphs of G which satisfy (2.2), (2.3), (2.5), (2.6), and (2.7) by Xm. Thus the formulation can be represented more compactly as:

$$mTSP-2 \quad \min \sum_{i=1}^{i=n} \left(c_{0i}x_{0i} + \sum_{j=1,j\neq i}^{j=n} c_{ij}x_{ij} \right)$$

$$\text{s.t.} \sum_{j=0,j\neq i}^{j=n} x_{ij} + \sum_{j=0,j\neq i}^{j=n} x_{ij} = 2 \qquad for \ i = 1,2,\cdots,n \tag{2.4}$$

$$x \in Xm, \qquad where \ x = (x_{ij})_{i,j\in N}$$

Denoting the set of all m-tours on G by M^m a still more compact formulation is obtained as follows:

$$mTSP-3 \quad \min \sum_{i=1}^{i=n} \left(c_{0i}x_{0i} + \sum_{j=1,j\neq i}^{j=n} c_{ij}x_{ij} \right)$$

s.t.

$$x \in M^m \tag{2.11}$$

A degree-constrained spanning tree with m components is a spanning tree on G in which the number of arcs incident to the base node is exactly

m [Glover & Klingman, 1975]. Defining an augmented degree-constrained spanning tree to be a degree-constrained spanning tree with m additional base arcs and the set of all augmented degree-constrained spanning trees to be Z^m, the multiple travelling salesmen problem can be stated as follows.

$$mTSP-4 \quad \min \sum_{i=1}^{i=n} \left(c_{0i}x_{0i} + \sum_{j=1,j\neq i}^{j=n} c_{ij}x_{ij} \right)$$

$$\text{s.t.} \sum_{j=0,j\neq i}^{j=n} x_{ij} + \sum_{j=0,j\neq i}^{j=n} x_{ji} = 2 \qquad \text{for } i = 1, 2, \cdots, n \tag{2.4}$$

$$x \in Z^m \tag{2.12}$$

III. RELAXATIONS OF mTSP AND SELECTION OF SUBGRAPH CONSTRUCTS

There is an immediate relaxation for the multiple travelling salesmen: A spanning tree problem with m additional base arcs is a relaxation obtained by removing the node-degree constraints, simply given by $\{\min c'x \mid x \in X^m\}$; and then the Lagrangean dual is obtained by dualizing the node-degree constraints.

Associating the Lagrange multiplier λ_i with the node-degree constraint for node i; and letting the corresponding vectors of Lagrange multiplier be given λ, we see that the Lagrangean relaxation with node-degree constraints dualized is given by:

$$\Re(\lambda, X^m) = \min_{x \in X^m} \left[\sum_{i=1}^{i=n} \left(c_{0i}x_{0i} + \sum_{j=1,j\neq i}^{j=n} c_{ij}x_{ij} \right) - \sum_{i=1}^{i=n} \lambda_i \left(\sum_{j=0,j\neq i}^{j=n} x_{ij} + \sum_{j=0,j\neq i}^{j=n} x_{ij} - 2 \right) \right] \tag{3.1}$$

The Lagrangean relaxation to the multiple travelling salesmen problem is given by $\Re(\lambda, X^m)$ and the Lagrangean dual simplifies to:

$$\underset{\lambda}{Max}\left[\min_{x\in X^m}\sum_{i=1}^{i=n}\left(c_{0i}x_{0i}+\sum_{j=1,j\neq i}^{j=n}c_{ij}x_{ij}\right)\right.$$
$$\left.+\sum_{i=1}^{i=n}\lambda_i\left(\sum_{j=0,j\neq i}^{j=n}x_{ij}+\sum_{j=0,j\neq i}^{j=n}x_{ij}-2\right)\right]$$

In solving the Lagrangean relaxation for any of the relaxations discussed in the previous paragraph, the choice of the graphical construct used to characterize the set X^m is central to development of an efficient solution procedure. Some routing problems have been solved by the use of graphical constructs very similar to the specifications for X^m: Held & Karp [Held & Karp 1970] introduced 1-trees for travelling salesman problems; m-trees are introduced in ([Ali & Kennington 1986]) for multiple travelling salesmen problems; augmented degree-constrained minimal spanning trees have been employed by Gavish & Srikanth [Gavish & Srikanth 1986] as a relaxation to multiple travelling salesmen problems. While both m-trees and augmented degree-constrained spanning trees meet the specifications of X^m, each of them allow the existence of immediate tours.

In this section we develop the <u>*connected m − tree*</u> construct for the multiple travelling salesmen problem. Using the set of connected m-trees for the set X^m instead of the set of m-trees or augmented degree-constrained spanning trees allows a tighter relaxation. In order to place the development into proper context, we first present a brief review of m-trees and augmented degree-constrained spanning trees.

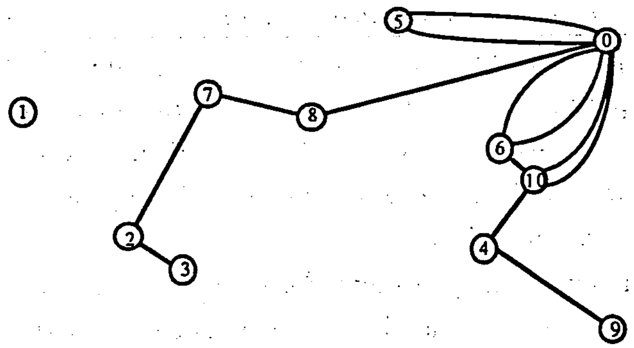
An <u>*m − tree*</u> on the graph G[N,E] consists of an m-forest on the node set N' and $2m$ base arcs connecting the base node 0 to non-base nodes, as illustrated by Figure 1(a). When a degree-constrained spanning tree is augmented by m more base arcs as illustrated by Figure 1(b), the constraint (2.3) in mTSP-1 is satisfied. Henceforth we refer to a degree-constrained spanning tree with m additional base arcs as an <u>*augmented degree − constrained spanning tree*</u>. Augmented degree-constrained spanning trees can be obtained by first obtaining a degree-constrained spanning tree with m base arcs and then selecting the cheapest additional m base arcs. Minimal weight degree-constrained spanning trees can be obtained by use of a quasi-greedy algorithm as developed in (Glover & Klingman, 1975).

The efficiency of the procedure for solution of the construct employed for X^m is important because it is used iteratively in the solution of the Lagrangean dual

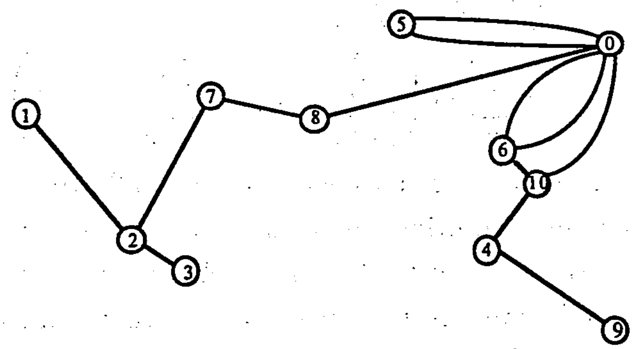
$$\underset{\lambda}{Max}\left[\min_{x \in X^m} \sum_{i=1}^{i=n}\left(c_{0i}x_{0i} + \sum_{j=1,j\neq i}^{j=n} c_{ij}x_{ij}\right) + \sum_{i=1}^{i=n}\lambda_i\left(\sum_{j=0,j\neq i}^{j=n} x_{ij} + \sum_{j=0,j\neq i}^{j=n} x_{ij} - 2\right)\right]$$

For multiple travelling salesmen problems allowing no existence of immediate tours in a solution, as such these two constructs do not, or can not provide the strongest relaxation. Modification of the augmented degree-constrained spanning tree procedure so that immediate tours are excluded is not straightforward. Without destroying the matroidal property, the definition of an m-tree can be modified so that immediate tours are not allowed. This is done simply by requiring that the $2m$ base arcs of the m-tree be distinct. However, since m-trees are not necessarily connected, such a modification still does not yield the strongest relaxation.

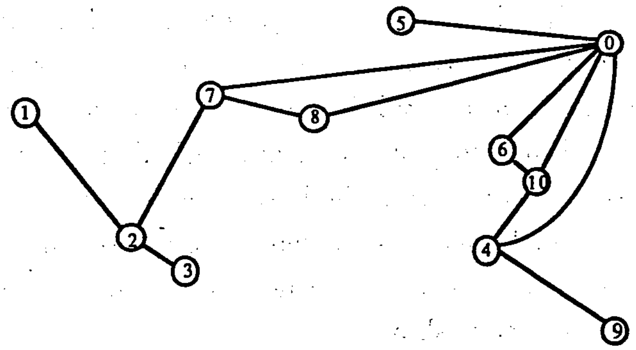
An <u>connected m − tree</u> on the graph G[N,E] is an m-tree in which each of $2m$ base arcs connecting the base node to non-base nodes are distinct and at least one base arc is incident to some node in each component of the m-forest on the node set N' formed by the non-base arcs. In Figure 1(c) a connected m-tree is shown.



(a) m-Tree Construct (m =3)



(b) Augmented Degree-Constrained Spanning Tree Construct (m =3)



(c) connected m-Tree Construct (m =3)

Figure 1: **Various Graphical Constructs for Multiple Travelling Salesman Problems.**

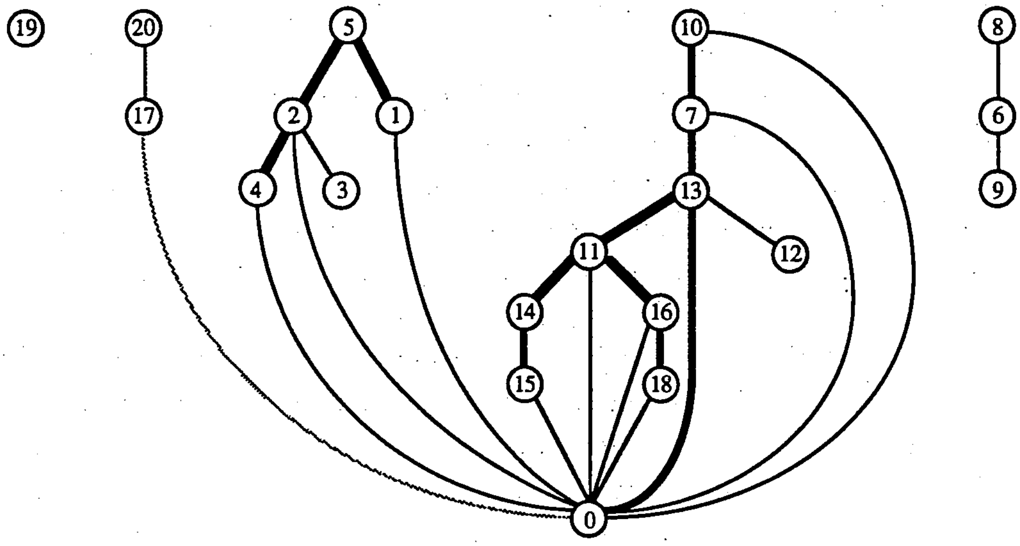
IV. QUASI-GREEDY ALGORITHM FOR MINIMUM COST CONNECTED m-TREES

Let X^m be the set of connected m-trees on the graph G. For each $x \in X^m$, let the corresponding arcs in the connected m-tree be given by the set χ. Before presenting the quasi-greedy algorithm for the solution of $\{\min cx \mid x \in X^m\}$, the following notation is introduced: Let H^m be the set of m-trees on G and, corresponding to $x \in H^m$, let be the sets of arcs of the m-tree. For the spanning forest on N' formed by the non-base arcs, the nodes in each component are given by subsets $N_k, k = 1, 2, \cdots, m$, of N'. For each component k, *link(k)* denotes the number of base arcs $e_{oj} \in \eta \mid j \in N_k$. A *connected component* of η is a component p, such that there exists at least one base arc $e_{oj} \in \eta$ (where $j \in N_p$). Note that for each connected component p, link(p) ≥ 0. An *isolated component*: of η is a component, q such that link(q) $= 0$. P(i,j) denotes the set of arcs of the m-tree η on the path from node i to node j where nodes i and j belong to the same component. The set $R_e = \{e_r \mid er \in P(i,j) for\ all\ i,j\ with\ e_{oj}, e_{oj} \in \eta\}$ denotes the set of *removable arcs* of the m-tree and the set of *attachable cross arcs* from an isolated component q to a connected component p is given by $C_q = \{e_c \mid e_c = e_{ij}\ with\ i \in N_q\ and\ j \in N_p\}$.

In much the same way that the quasi-greedy algorithm is designed for degree-constrained spanning trees, the algorithm for obtaining a connected m-tree proceeds by first obtaining the solution η to $\{\min cx \mid x \in H^m\}$. Clearly if no component is isolated, then η is the solution to $\{\min cx \mid x \in X^m\}$. Otherwise each isolated component q can be connected to the base node by one of the following permissible exchanges. A *permissible exchange* defines an exchange between an arc $\in \eta$ and another arc $\in \eta$ which serves to connect the component q to the base node either directly via a base arc or indirectly via a non-base arc. There are three types of permissible exchanges:

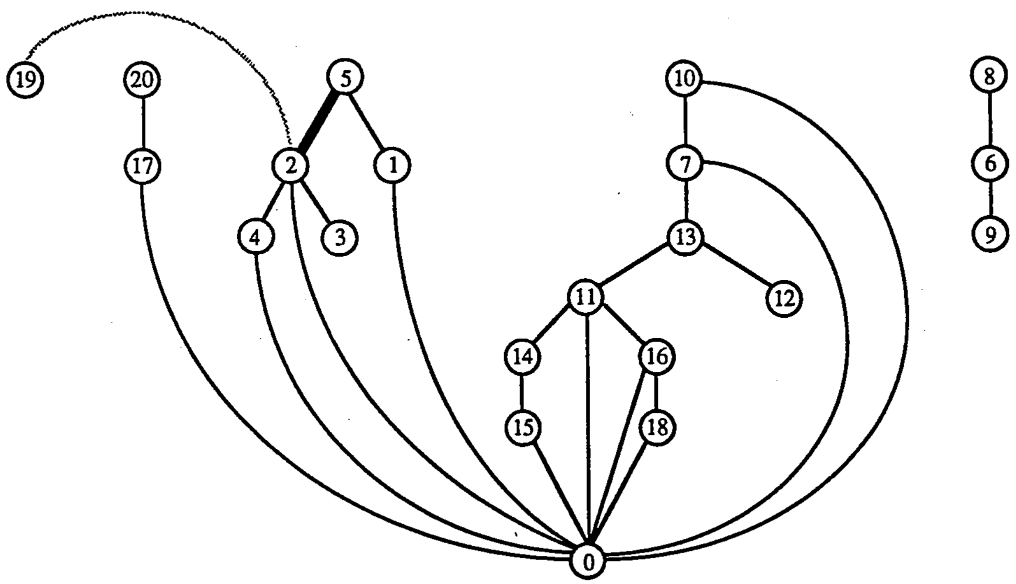
1. The exchange between an $e_{oi} \in \chi$ and an $e_{oj} \notin \chi$ wherenode i is in a component p with link(p) ≥ 2, and j is anode in an isolated component q as illustrate fromFigure 2(a) to Figure 2(b).

2. The exchange between an attachable arc e_c in C_q and aremovable arc in R_e where q is an isolated component asshown from Figure 2(b) to Figure 2(c).
3. A two-step exchange consisting of: (a) the exchangebetween an $e_{oi} \in \eta$ and an $e_{oj} \notin \eta$, where node i is ina component p with link(p) ≥ 2, and j is a node also ina connected component p'(p may be the same as p'); and(b) the exchange between an attachable arc e_c in C_q and one of the new removable arcs in R_e by the introductionof a new base arc e_{oj} in the previous step, where q is anisolated component. Such a two-step exchange is shownfrom Figure 2(c) to Figure 2(d).

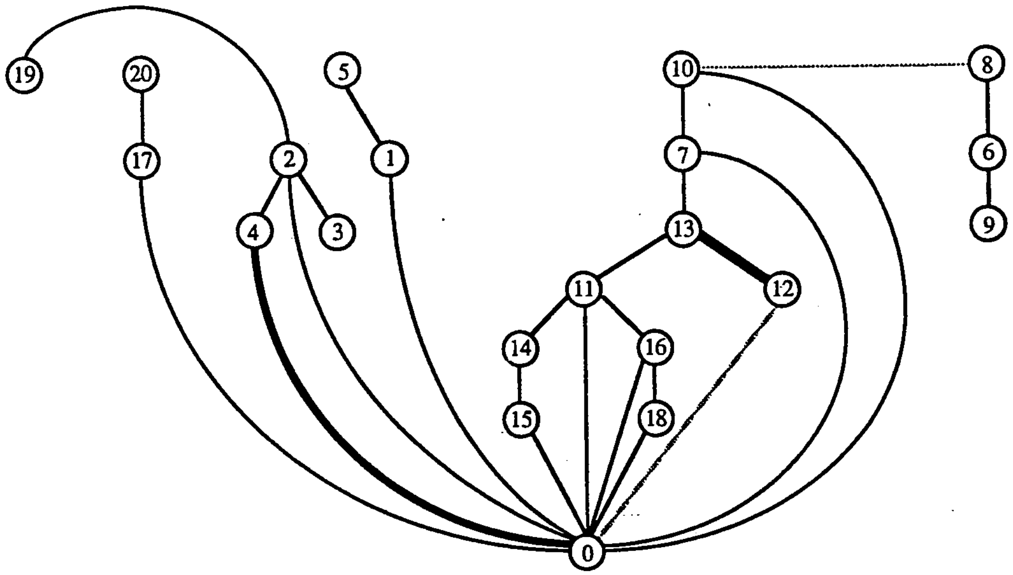


(a) Minimal m-tree (20 nodes, m=5) with 3 isolated components

Figure 2 : Illustration of Construction of Connected m-Tree. (continued)
(———— entering arc ; ▬▬▬▬ leaving arc)

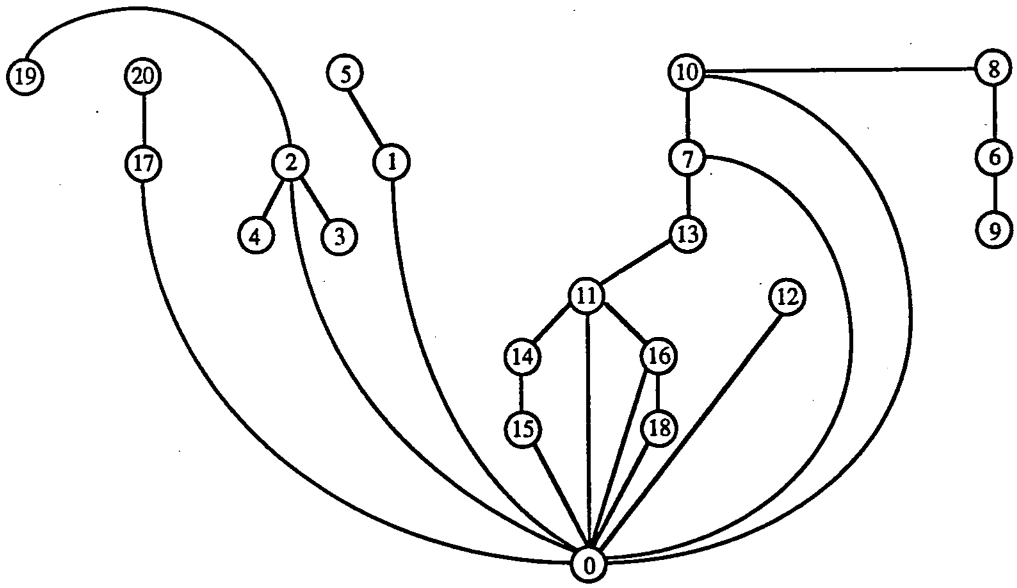


(b) Intermediate m-tree via type-1 permissible exchange



(c) Intermediate m-tree via type-2 permissible exchange

Figure 2 : Illustration of Construction of Connected *m*-Tree.
(——— entering arc ; ▬▬▬ leaving arc)



(d) Intermediate m-tree via type-3 permissible exchange

Figure 2 : Illustration of Construction of Connected *m*-Tree. (continued)
(——— entering arc ; ━━━ leaving arc)

A type-1 exchange directly connects an isolated component to the base node without destroying the connectivity of existing connected components of η. A type-2 exchange removes an element of the set of removable arcs which belongs to the path P(s,t) where s and t are nodes of some connected component p. When this arc is removed, the component p is split into two connected sub-components. When an attachable arc of the isolated component q is introduced, it is connected to one of these connected component. In either of the these two exchanges the number of isolated components reduces by 1. A type-3 exchange is a combination of type-1 and type-2 exchanges. First, a base arc of a connected component is removed and replaced by another base arc to a connected component. By such an exchange, the set of removable arcs changes. Then a type-2 exchange is performed whereby an attachable arc is introduced and one of the newly created removable arcs is removed.

Algorithm: Minimum Cost Connected m-Tree

Step 0 Let η solve $\{\min cx \mid x \in H^m\}$
k = number of isolated components in η;

for t = 1,· · ·, k, do

begin

$\alpha_{out} \leftarrow -\infty, \alpha_{in} \leftarrow \infty;$

$\beta_{in} \leftarrow \infty, \beta_{out} \leftarrow -\infty;$

$Z_{\beta-2} \leftarrow \infty;$

Step 1 For each connected component p, find link(p);

For each isolated component q (non-base node set N_q), do

begin

$\alpha_{in} \leftarrow (\alpha_{in}, \min_{i \in N_q}\{c(e_{0i})\});$

$\beta_{in} \leftarrow (\beta in, \min_{e_j \in C_q}\{c(e_i)\});$

end ;

For each connected component p, do

begin

If link(p)$\geq 2, \alpha_{out} \leftarrow \max(\alpha_{out}, \max_{i \in N_p, e_{0i} \in \chi}\{c(e_{0i})\});$

Find removable arcs e_r in E_p ,

$\beta_{out} \leftarrow \max(\beta_{out}, \max_{all\ e_r}\{c(e_r)\});$

end;

For each $e_{0i} \in \chi, i \in$ any N^p, do

begin

Find additional removable arcs of $e_{0i}e_r$ in E_p;

If $(\beta_{in}-, \max_{all\ e_r}\{c(e_r)\} + c(e_{0i}) - \alpha_{out}) < z_{\beta-2}$, then

$z_{\beta-2} \leftarrow (\beta_{in}-, \max_{all\ e_r^i}\{c(e_r^i)\} + c(e_{0i} - \alpha_{out})$

end

end;

Each time a pivot is performed, the number of isolated components is reduced by 1. Thus if there are k ($k < m$) isolated components, exactly k permissible exchanges are performed in constructing a minimum weight connected m-tree. Since the cheapest of permissible exchanges is chosen, validity of the procedure is ensured.

V. COMPUTATIONAL RESULTS AND CONCLUSIONS

Table 1 reports computational testing with four graphical constructs which can be used for the definition of X^m in the model for the multiple travelling salesmen problem. The objective function values obtained when the problem $\{\min cx \mid x \in X^m\}$ is solved are tabulated where X^m is defined to be:(1) the set of augmented degree-constrained spanning trees which allow immediate tours and are connected subgraphs; (2) the set of m-trees which allow immediate tours and are not necessarily connected subgraphs (3) the set of m-trees which do not allow immediate tours and are not necessarily connected; (4) the set of connected m-trees which are connected by definition. The comparison of the objective function values is tested on randomly generated dense graphs with 20 and 50 nodes. The m-tree relaxations with and without immediate tours are solved using a greedy algorithm while the other two employ quasi-greedy algorithms. The connected m-tree provides the strongest relaxation for the balanced multiple travelling salesmen problem as evident from the objective function values in the table. For the five 20-node problems, the connected m-tree has an objective function value which is on the average 8% higher than that for the augmented degree-constrained spanning tree, 22% higher than for an m-tree with immediate tours, and 7% higher than for an m-tree without immediate tours. For the five 50-node problems, the connected m-tree has an objective function value which is on the average 4% higher than that for the augmented degree-constrained spanning tree, 14% higher than for an m-tree with immediate tours, and 8% higher than for an m-tree without immediate tours.

Table 1 : Comparison of Relaxations to Multiple Travelling Salesman Problems

ID	n	m	A-DCST	m-Tree(1)	m-Tree(2)	C-m-Tree
A	20	4	459.95	448.98	475.47	480.34
B	20	4	378.65	306.67	370.16	411.45
C	20	4	374.44	354.05	395.59	403.34
D	20	4	284.37	245.63	281.73	314.20
E	20	4	343.06	284.59	337.65	367.27
A	50	5	1490.54	1377.30	1475.90	1579.17
B	50	5	1483.59	1349.35	1429.24	1539.41
C	50	5	1472.98	1387.41	1457.40	1531.81
D	50	5	1368.21	1205.42	1262.41	1413.84
E	50	5	1301.26	1180.04	1248.10	1339.22

Note :

1. A-DCST : Augmented degree-constrained spanning tree with immediate tours.
2. m-tree(1) : m-tree with immediate tours.
3. m-tree(2) : m-tree without immediate tours.
4. C-m-tree : Connected m-tree.

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多旅行銷售員問題的
多連接樹鬆弛法

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摘 要

對於較大型之多旅行銷售員問題，目前合理之求最佳解的方式乃是分支限界法與鬆弛法並用。前者用來取得上限，後者則是用來求取下限。在求單一旅行銷售員問題之最佳解上，Held與Karp兩人曾提出『多一支之擴張最小生成樹』爲其鬆弛之基礎模式，得到極佳之效果。由於此種破題之方式不但能保留問題之圖像本質，所產生之下限又相當強；若干研究者乃根據類似之思考方式對於多旅行銷售員問題提出「多叢樹模型」、「次數限制擴張生成樹模型」來做爲鬆弛法之基本模型。本論文之特點與貢獻在於提出一種比過去之研究在圖形結構更完整，在運算結果更強之新鬆弛模型，稱之爲多連接樹鬆弛法，以做爲解多旅行銷售員問題時之下限模式。

關鍵詞：圖像，旅行銷售員問題，解法，鬆弛法