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TECHNOLOGY BREAKTHROUGH WITHIN EQUIPMENT REPLACEMENT MODELS UNDER MARKOVIAN DETERIORATION

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ABSTRACT

Extending Nair (1989) study, this paper deals with the equipment replacement under Markovian deterioration and technological obsolescence. We assume that the accumulated impact of the new knowledge results in a single technology breakthrough and consequently improved equipment is available on the market. A crucial issue in management is to decide for each period whether to keep using the existing equipment or to procure new equipment so as to maximize the total expected discounted reward in the infinite horizon. Two functions have to be taken into consideration for deciding on an optimal policy, namely, reward functions and conditional probability distribution function of technology breakthrough. The reward functions depend on the state and the age of equipment, and the conditional probability distribution of technology breakthrough is based on the shock model of Aven and Gaarder (1987) with revision. We then establish the recursive formula by using stochastic control theory. We prove that the conditional probability distribution of technology breakthrough is constant after technology forecast period under some reasonable conditions. We also show that the optimal policy has a special control limit structure form of equipment state and age. Furthermore, we prove that the optimal value function is nondecreasing when technology breakthrough rates are nondecreasing in time. Ultimately, the existence of a forecast horizon for optimal replacement policy is demonstrated.

Key Words: Markovian deterioration, Technology breakthrough, Control limit, Optimal replacement policy, Forecast horizon.

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I. INTRODUCTION

Advanced technology shortens the useful life of sophisticated products such as electronics and computers. With the technology breakthroughs on the market, decision makers of manufacturing and service firms have to decide whether to keep the existing equipment or to replace it with a better technology.

Relevant studies of economic life of equipment originated from Preinreich (1940) and Terborgh (1949). Bellman (1957) brings forward the body of literature addressing the model on replacement problem under Markovian deterioration. Later, Dreyfus (1960), Derman (1963) inherits Bellman's study and come up with stochastic deterioration of the equipment (for surveys, see Barlow et al. (1965), McCall (1965), Pierskalla and Voelker (1976), Sherif and Smith (1981)).

In terms of the optimal replacement, most of the above models ignore the prospect of rapid technological improvement. In this article we are interested in the problem of replacing a system subject to deterioration with technology improvements. Models of this kind were analyzed by Sethi and Morton (1972), Sethi (1973), Elton and Gruber (1976), Sethi and Chand (1979), Chand and Sethi (1982), Oakford et al. (1984), Bean et al. (1985), Kusaka(1986), Goldstein et al. (1986,1988), Bhaskaran and Sethi (1987), Hopp and Nair (1991), Nair and Hopp (1992), and Bylka et al. (1992). They arrived at the conclusion that forecast horizons can solve the infinite horizon problem. The forecast horizon is defined as the minimum number of periods of forecasted information required to guarantee the initial optimal decision, regardless of forecasts in later periods. None of these papers, however, consider Markovian deterioration with stochastic technological forecasts. Only Nair (1989) addressed the equipment replacement decisions due to technological obsolescence under Markovian deterioration. He proves that control limit structure exists for the optimal policy. Under certain conditions the control limit is nondecreasing with increasing rate of technological change. Besides, he investigates the existence of the forecast horizon to optimize the infinite horizon problem.

In Nair's study, however, the equipment age is not taken into account, neither is the probability distribution function of technology breakthrough. In our study, we consider all the above factors, especially the equipment age and the impact of the new knowledge. We follow the shock model in Aven and Gaarder (1987) in explaining the impact of the new knowledge and technology breakthrough as a result of shocks, different from the original concept of in which the system would fail at any point in time due to the accumulated damages. Finally, based on some reasonable conditions, we then obtain a much more feasible optimal replacement policy for management applications.

II. MODEL DESCRIPTION

We consider a firm owning two pieces of equipment of the present level of technology. We also assume that at most one new technology may appear in the future. Usually, the equipment is undergoing deterioration over time and can be observed completely. The problem of the firm is trying to decide whether to keep existing equipment or to replace it with an equipment of better technology currently available on the market. We then consider the case where the equipment replacement action must be done instantaneously at the beginning of the period. The firm's objective is to find a sequence of replacement actions so as to maximize the total expected discounted reward over the infinite horizon. Our notation are stated below.

X: equipment set, X = {0,1} where the performance of equipment "1" is better than that of equipment "0".

Y : equipment set, Y = {0,1,2} where the performance of equipment "2" is better than that of equipment "1".

D_g : state space for equipment g, $D_g = \{1,...,m_g\}$, $g \in Y$, $m_g \in N$, $m < \infty$, where index 1 represents the "new" equipment and m_g a "broken down" equipment.

k_g : the action "keep using existing equipment g".

R_q : the action "replace with new equipment q".

A(g,n,i;X): action space for state (g,n,i;X), and let A(g,n,i;X) = $\{k_g, R_q, q \in X\}$; similarly, A(g,n,i;Y) for state (g,n,i;Y), and let A(g,n,i;Y) = $\{K_g, R_q; q \in Y\}$.

ρ(g,n,i;a) : bounded immediate reward for action a $\in$ {A(g,n,i;X) $\cup$ A(g,n,i;Y)}.

$P_{t+1}(Y|X)$: conditional probability of technology breakthroughs where Y is available at the beginning of period t+1, given that technology breakthroughs in X are available at the begin ning of period t.

P^a_{gn} : Markovian deterioration matrix of equipment g at age n under action "a", and let $P^a_{gn}=[p^a_{gn}(i,j)]$, where $p^a_{gn}(i,j)$ denotes the transition probability if equipment g at age n in state i and action a is performed, then it deteriorates to state j.

c_i : procuring cost of equipment i, i =0,1,2 ,where $c_0 \le c_1 \le c_2$.

β : one period discount factor, $0<\beta<1$.

f_t^T(g,n,i;X): optimal value functions in periods t through T, discounted back to the beginning of period t, given the state at t is (g,n,i;X); similarly f_t^T(g,n,i;Y) denotes at the state (g,n,i;Y) .

π_t^T(g,n,i;X): optimal action in state (g,n,i;X); similarly, π_t^T(g,n,i;Y) denotes at the state (g,n,i;Y).

III. ASSUMPTIONS

To characterize the structure of the optimal policy, we will make use of the following assumptions.

(A1) The Markovian deterioration matrix $P^a_{gn}=[p^a_{gn}(i,j)]$ is assumed to have $p^a_{gn}(i,i)<1$ for all i$\in D_g-\{m_g\}$, a$\in$\{A(g,n,i;X)$\cup$A(g,n,i;Y)\}, and

$\sum_{j\ge k} p^a_{gn}(i,j)$ is nondecreasing in i for all $k\in D_g$.

Assumption (1) assures that the underlying Markov chain is upper triangular and has the IFR property (Derman (1963)). i.e., If we keep using existing equipment, then the equipment will acceleratly deteriorate in its worse state.

(A2) Let $\ge_{st}$ represent "Stochastic Dominance Partial Order", if $P^a_{gn}\ge_{st}P^a_{gm}$ and

$m \geq n$, then the following conditions are equivalent:

$$\text{(i)} \quad \sum_{j \leq k} p^a_{gn}(i,j) \geq \sum_{j \leq k} p^a_{gm}(i,j)$$

and

$$\text{(ii)} \sum_{j \geq k} p^a_{gm}(i,j) \geq \sum_{j \geq k} p^a_{gn}(i,j) \quad \text{for } i,j,k \in D_g .$$

Assumption (2) implies that the older equipment tends to deteriorate much easier than that of younger ones.

(A3) For any fixed a, we assume
a) ρ(g,n,i;a) is nonincreasing in i for all n.
b) ρ(g,n,i;a) is nonincreasing in n for all i.

Assumption (3a) denotes that equipment states with lower index gain a higher immediate reward. Similarly, assumption (3b) states that ages with lower index gain a higher reward.

(A4) The better the equipment technology is, the higher the maximal total expected discounted reward will be.

Assumption (4) claims that we always replace the existing equipment with the updated equipment on the market, i.e. the equipment will be more dominant when it first appears on the market.

(A5) For $m \geq n$ and any action a, there exists a state $k \in D_g$ such that

$$k = \min_{l \in D_g} \{l : \theta^a_{nm}(i,j) \geq 0, j < l \text{ and } \theta^a_{nm}(i,j) \leq 0, j \geq l\}$$

where

$$\theta^a_{nm}(i,j) = p^a_{gn}(i,j) - p^a_{gm}(i,j)$$

Assumption(5) addresses that there exists a certain state with the age of the equipment considered: older equipment tends to deteriorate faster than that of the younger ones. In fact this assumption is similar to that of the assumption (2) but with stronger means.

IV. MODEL FORMULATION

First of all, we need to construct information state space (S) in terms of g, n, i, X or Y as follow:

$$S = \{ \{g,n,i,;X\} \cup \{g,n,i;Y\} \}.$$

For notation convenience, we will let $\overline{X} = \{X,Y\}$, where $X=\{0,1\}$, $Y=\{0,1,2\}$. The optimal value function over the finite horizon given state (g,n,i;X) at time t can be written as follows:

$$f_t^T(g,n,i;X)$$

$$= \max_{a \in A(g,n,i;X)} \rho(g,n,i;a) + \beta \sum_{X' \in \overline{X}} p_{t+1}(X'|X)\{\sum_{j \geq 0} p_{gn}^a(i,j) f_{t+1}^T(g,n,i;X')\} \quad (1)$$

for all $(g,n,i;X) \in S$.

We define

$$\pi_t^T(g,n,i;X)$$

$$= \{\bar{a} \in A(g,n,i;X): \rho(g,n,i;\bar{a}) + \beta \sum_{X' \in \overline{X}} p_{t+1}(X'|X)\{\sum_{j \geq 0} p_{gn}^{\bar{a}}(i,j) f_{t+1}^T(g,n,i;X')\}$$

$$\geq \rho(g,n,i;a) + \beta \sum_{X' \in \overline{X}} p_{t+1}(X'|X)\{\sum_{j \geq 0} p_{gn}^a(i,j) f_{t+1}^T(g,n,i;X')\}$$

for $a, \bar{a} \in A(g,n,i;X)\}$,

where

$$\rho(g,n,i;a) = \begin{cases} -c_q + r(q,0,1), & a = R_q \\ r(1,0,i) \quad , & a = K_g \end{cases}, \; g,q \in X.$$

and

$$p_{gn}^{a}(i,j)=\begin{cases}p_{q0}(i,j), & a=R_q\\ p_{gn}(i,j), & a=K_g\ , g,q\in X.\end{cases}$$

Similarly, f_t^T(g,n,i;Y) has the form same as above.

To develop the probability distribution function of technology breakthrough, we need the following:

(A6) The magnitude of new knowledge impact contribution index is independent of the occurring time.

The new knowledge impact model can be established by referring to the shock model of Aven and Gaarder (1987) with revision, as follows:

Let (Ω,Σ,P) be a complete probability space, and $\{F_t$, t = 0,1,2,...} be a nondecreasing family of sub$-\sigma-$field of Σ such that F_0 includes all null sets of Σ. The $\sigma-$field F_t represents the set of information without the occurrence of technology breakthrough up to time t. We also let V be the appearing time of technology breakthrough, then event { V > t } $\in F_t$, t = 0,1,2,$\cdots$.

Under the assumption where technology breakthrough can possibly occur at the new knowledge impact, we define

$$W_t=\begin{cases}1, & \text{new knowledge impact occurs at time t.}\\ 0, & \text{otherwise}\end{cases}$$

$$W_0=0$$

and let

$$U_t = u \cdot 1_{\{W_t=1\}} = \begin{cases} u, & W_t = 1 \\ 0, & W_t = 0 \end{cases}$$

$$U_0 = 0, u \in [0,1]$$

where

U_t : contribution index of new knowledge impact occurring at time t.

Then

$$P\{W_{t+1} = 1 | F_t\} = a_t\ , \quad t < V_t$$

where

a_t: the conditional probability that a new knowledge impact occurring at time t+1, given that the technology breakthrough doesn't appear up to time t.

Hence

$$\begin{aligned} & p\{U_{t+1} \le u | F_t \vee \sigma(W_{t+1})\} \\ & = W_{t+1} H_t(u) + (1 - W_{t+1}) 1\{u > 0\} \\ & = \begin{cases} H_t(u), & W_{t+1} = 1 \\ 1 \quad , & W_{t+1} = 0, t < v \end{cases} \end{aligned}$$

where

$H_t(u)$: the conditional probability distribution of contribution index u for the new knowledge impact occurs at time t+1, given that the technology breakthrough doesn't appear up to time t.

Moreover

$$
\begin{aligned}
& p\{V = t+1 | F_t \vee \sigma(W_{t+1})\} \\
& = W_{t+1} l_t(u) + (1 - W_{t+1}) g_t \\
& = \begin{cases} l_t(u), & W_{t+1} = 1 \\ g_t \quad , & W_{t+1} = 0 \end{cases}
\end{aligned}
$$

where

$l_t(u)$: the conditional probability that the technology breakthrough appears at time t+1, given that technology breakthrough doesn't appear up to time t, whereas let the new knowledge impact contribution index be u at time t+1.

g_t: the conditional probability that the technology breakthrough appears at time t+1, given that technology breakthrough doesn't appear up to time t, whereas the new knowledge impact doesn't occur at time t+1.

V. STRUCTURAL RESULTS

Before showing the main results, we need the following technical lemma to develop the conditional probability distribution of technology breakthrough.

Lemma 3.1: Under assumption (6), we have

$$P_{t+1}(Y|X) = \alpha_t,\ t = 1,2,...$$

where

$$\alpha_t = a_t \int_0^1 l_t(u) H_t(du) + (1 - a_t) g_t$$

Proof:

$$
\begin{aligned}
&P_{t+1}(Y|X)\\
&=P\{V=t+1\}\\
&=E[P\{V=t+1|F_t\vee\sigma(W_{t+1},U_{t+1})\}|F_t]\\
&=E[W_{t+1}l_t(U_{t+1})|+(1-W_{t+1})g_t|F_t]\\
&=E[W_{t+1}E[l_t(U_{t+1})|F_t\vee\sigma(W_{t+1})]|F_t]+g_tP\{W_{t+1}=0|F_t\}\\
&=E[W_{t+1}\int_0^1 l_t(u)H_t(du)|F_t]+(1-a_t)g_t\\
&=a_t\int_0^1 l_t(u)H_t(du)+(1-a_t)g_t\ . \qquad \square
\end{aligned}
$$

Next, by using Lemma 3.1 and assumption (4), recursion (1) can be rewritten as follows for each $g\in X$ and $t=0,1,2,\ldots,T\text{-}1$:

$$
f_t^T(g,n,i;X)=\max\begin{cases} R_1:\ -c_1+f_t^T(i,0,1;X)\\ K_g:\ r(g,n,i)+\beta\ [(1-\alpha_t)\sum\limits_{j\geq i}p_{gn}(i,j)f_{t+1}^T(g,n+1,j;X)\\ \qquad +\alpha_t\sum\limits_{j\geq i}p_{gn}(i,j)f_{t+1}^T(g,n+1,j;Y)]\end{cases} \tag{2}
$$

where

$$
f_t^T(1,0,1;X)=r(1,0,1)+\beta[(1-\alpha_t)\sum_{j\geq i}p_{10}(1,j)\ f_{t+1}^T(1,1,j;X)
$$

$$
+\ \alpha_t\ \sum_{j\geq 1}p_{10}(1,j)f_{t+1}^T(1,1,j;Y)]
$$

Similarly, $f_t^T(g,n,i;Y)$ can be presented as follows for each $g\in Y$:

$$
f_t^T(g,n,i;Y)=\max\begin{cases} R_2:-c_2+f_t^T(2,0,1;Y)\\ K_g:r(g,n,i)+\beta\,[\sum\limits_{j\geq i}p_{gn}(i,j)f_{t+1}^T(g,n+1,j;Y)\end{cases} \tag{3}
$$

where

$$f_t^T(2,0,1;Y) = r(2,0,1) + \beta[\sum_{j\geq 1} p_{gn}(1,j) f_{t+1}^T(2,1,j;Y)]$$

Theorem 3.1: If a_t, g_t, l_t (u) and H_t (u) are bounded and nondecreasing in t, then there exists $t^* \in N$, such that $P_{t+1}(Y|X)$ is a constant for all $t \geq t^*$.

Proof: By Lemma 3.1, we know that

$$P_{t+1}(Y|X) = a_t \int_0^1 l_t(u) H_t(du) + (1-a_t) g_t$$

Moreover, by monotone convergence theorem, it is easy to show that there exists $t^* \in N$, such that a_t, g_t converge at some constants, and l_t (u), H_t (u) converge at l(u) and H(u) for all $t \geq t^*$ respectively.

Hence, these results complete the proof. □

Lemma 3.2: Under assumptions (1) and (2), if $f_n(j)$ be nonincreasing function in j for any fixed n, then $\sum_{j\geq i} p_{gn}^a(i,j) f_n(j)$ is nonincreasing in i.

Proof: It is done by Derman's Lemma (1963). □

Lemma 3.3: Under assumptions (2) and (5), if $f_n(j)$ be nonincreasing function in n for any fixed j and be nonincreasing in j for any fixed n, then $\sum_{j\geq i} p_{gn}^a(i,j) f_n(j)$ is nonincreasing in n.

Proof:

(i) If $a = R_q$, then $\sum_{j\geq i} p_{gn}^a(i,j) = \sum_{j\geq 1} p_{qo}(i,j)$ satisfied the result.

(ii) If $a = k_g$, then $\sum_{j\geq i} p_{gn}^a(i,j) = \sum_{j\geq 1} p_{gn}(i,j)$. Putting $m \geq n$, by using assumption (5), there exists k such that $i \le k \le m_g$, and $p_{gn}(i,j) - p_{gm}(i,j) \ge 0$ for j < k; $p_{gn}(i,j) - p_{gm}(i,j) \le 0$ for $j \ge k$.

Furthermore, with the assumption (2) along with above k, we have

$$\sum_{j \ge i} p_{gn}(i,j) f_n(j) - \sum_{j \ge i} p_{gm}(i,j) f_m(j)$$

$$= \sum_{j<k} p_{gn}(i,j) f_n(j) - \sum_{j<k} p_{gm}(i,j) f_m(j) + \sum_{j \ge k} p_{gn}(i,j) f_n(j) - \sum_{j \ge k} p_{gm}(i,j) f_m(j)$$

$$\ge \sum_{j<k} p_{gn}(i,j) f_n(j) - \sum_{j<k} p_{gm}(i,j) f_n(j) + \sum_{j \ge k} p_{gn}(i,j) f_n(j) - \sum_{j \ge k} p_{gm}(i,j) f_n(j)$$

$$= \sum_{j<k} [\, p_{gn}(i,j) - p_{gm}(i,j)] f_n(j) + \sum_{j \ge k} [\, p_{gn}(i,j) - p_{gm}(i,j) \,] f_n(j)$$

$$\ge \sum_{j<k} [\, p_{gn}(i,j) - p_{gm}(i,j)] f_n(k) + \sum_{j \ge k} [\, p_{gn}(i,j) - p_{gm}(i,j) \,] f_n(k)$$

$$= \{ [\sum_{j<k} p_{gn}(i,j) + \sum_{j \ge k} p_{gn}(i,j)] - [\sum_{j<k} p_{gm}(i,j) + \sum_{j \ge k} p_{gm}(i,j)] \} f_n(k)$$

$$= 0$$

Therefore, from cases (i) and (ii), the lemma is proved. □

We can achieve stronger structure conclusions, if we set the boundary conditions as below:

$$f_T^T(g,n,i;X) = L(g,n,i;X), \quad g = 0,1$$

$$f_T^T(g,n,i;Y) = L(g,n,i;Y), \quad g = 0,1,2$$

It is obvious that all boundary conditions above are non-negatives. From here and by Lemma 3.2, we get the following.

Lemma 3.4: Under assumptions (1)-(6),

a) If $L(g,n,i;Y)$ is nonincreasing in i for any fixed n and $g \in Y$, then $f_t^T(g,n,i;Y)$ is nonincreasing in i.

b) If $L(g,n,i;Y)$ is nonincreasing in n for any fixed $i \in D_g$ and $g \in Y$, then $f_t^T(g,n,i;Y)$ is nonincreasing in n.

Proof of (a): From recursion (3), and Lemma 3.2, it is easy to see that $f_{T-1}^T(g,n,i;Y)$ is nonincreasing in i.

Now, we suppose that $f_{t+1}^T(g,n,i;Y)$ being nonincreasing in i holds, and we show that this implies that $f_t^T(g,n,i;Y)$ is nonincreasing in i. Since $\rho(g,n,i;a)$ is nonincreasing in i, the result follows from Lemma 3.2 by induction.

Proof of (b): Similarly, it is shown that $f_t^T(g,n,i;Y)$ is nonincreasing in n directly with Lemma 3.3. □

Lemma 3.5: Under assumptions (1)-(6),

a) If $L(g,n,i;X)$ is nonincreasing in i for any fixed n and $g \in X$, then $f_t^T(g,n,i;X)$ is nonincreasing in i.

b) If $L(g,n,i;X)$ is nonincreasing in n for any fixed $i \in D_g$ and $g \in X$, then $f_t^T(g,n,i;X)$ is nonincreasing in n.

Proof: From recursion (2), it has the identical approach on the Lemma 3.4, the proof is done. □

To further characterize the value function and optimal policy for recursion (2) and (3) under infinite horizon, we define

$$\lim_{T\to\infty} f_t^T(g,n,i;X) = f_t(g,n,i;X) \tag{4}$$

$$\lim_{T\to\infty} f_t^T(g,n,i;Y) = f(g,n,i;Y) \tag{5}$$

We are particularly interested in the infinite horizon optimal action $\pi_t(g,n,i;X)$ and $\pi(g,n,i;Y)$ respectively. To do this, let

$$\Delta_t^T(g,n,i;X)$$

$$=-c_1+[r(1,0,1)-r(g,n,i)]+\beta(1-\alpha_t)[\sum_{j\geq 1} p_{10}(1,j)f_{t+1}^T(1,1,j;X)$$

$$-\sum_{j\geq i} p_{gn}(i,j)f_{t+1}^T(g,n+1,j;X)]+\beta\alpha_t[\sum_{j\geq 1} p_{10}(1,j)f_{t+1}^T(1,1,j,Y)$$

$$-\sum_{j\geq i} p_{gn}(i,j)f_{t+1}^T(g,n+1,j;Y)]$$

And let $T\to\infty$, we then have

$$\Delta_t^T(g,n,i;X)=\Delta_t(g,n,i;X)$$

Similarly,

$$\Delta_t^T(g,n,i;Y)=\Delta(g,n,i;Y)$$

where

$$\Delta(g,n,i;y)=-c_2+[r(2,0,1)-r(g,n,i)]+\beta\ [\sum_{j\geq 1} p_{20}(1,j)f(2,1,j;Y)$$

$$-\sum_{j\geq i} p_{gn}(i,j)f(g,n+1,j;Y)]$$

We also define the control limits of equipment state and age,

$$i_t^*(g)=\min\{\text{ i : } \Delta_t(g,n,i;X)>0 \text{ for all n} \in \text{N}\}$$

$$n_t^*(g,i)=\min\{\text{ n : } \Delta_t(g,n,i;X)>0 \text{ for all i} < i_t^*(g)\}.$$

and

$$i^*(g) = \min\{ i : \Delta_t(g,n,i;Y) > 0 \text{ for all } n \in N \}$$
$$n^*(g,i) = \min\{ n : \Delta_t(g,n,i;Y) > 0 \text{ for all } i < i^*(g) \}.$$

From the above conditions (4) and (5), and with Lemmas 3.1 to 3.4, we can get the main theorems.

Theorem 3.2:

a) In state $(g,n,i;X)$ for each period t, there exists an integer $i_t^*(g) \in D_g$ such that $\pi_t\ (g,n,i;X) = R_1$ if $i \geq i_t^*(g)$ for all n.

b) If $i < i_t^*(g)$, then there exists $n_t^*(g,i) \in N$ such that $\pi_t(g,n,i;X) = R_1$ for $n \geq n_t^*(g,i)$ and $\pi_t(g,n,i;X) = k_g$ for all $n < n_t^*(g,i)$.

Proof of (a): It is sufficient to prove that $\Delta_t(g,n,i;X)$ is nondecreasing in i for all fixed n and $\Delta_t(g,n,i;X)$ is nondecreasing in n for all fixed i. Hence, using Lemmas 3.2 to 3.4, and assumption (3), it follows that $\Delta_t(g,n,i;X)$ is nondecreasing in i for any fixed n and is nondecreasing in n for any fixed i.

Thus, there exists a control limit $i_t^*(g)$ such that if $i \geq i_t^*(g)$ then $\pi_t(g,n,i;X) = R_1$ for all n. The result of (a) is proved .

Proof of (b): From Lemmas 3.3 and 3.4, and assumption (3), we can easily show that $\Delta_t(g,n,i;X)$ is nondecreasing in n for each $i < i_t^*(g)$.

Hence, there exists a control limit $n_t^*(g,i)$ such that if $n \geq n_t^*(g,i)$ then $\Delta_t(g,n,i;X)>0$ and $\pi_t(g,n,i;X) = R_1$, and if $n<n_t^*(g,i)$ then $\Delta_t(g,n,i;X) \leq 0$ and $\pi_t(g,n,i;X) = k_g$. Leading to the result of (b). □

Theorem 3.3:

a) In state $(g,n,i;Y)$, there exists an integer $i^*(g)$, such that $\pi(g,n,i;Y) = R_2$ if $i \geq i^*(g)$ for all n.

b) If $i < i^*(g)$, then there exists $n^*(g,i) \in N$ such that $\pi(g,n,i;Y) = R_2$ for all $n \geq n^*(g,i)$ and $\pi(g,n,i;Y) = k_g$ for all $n < n^*(g,i)$.

Proof: By the same token, the result of (a) and (b) is proved. □

To strengthen the properties of the optimal policy structure in Theorems 3.2 and 3.3, we need the following:

Corollary 3.1:

a) If $i < i_t^*(g)$ then $n_t^*(g,i)$ is nonincreasing in i.

b) If $i < i^*(g)$then $n^*(g,i)$ is nonincreasing in i.

Proof of (a): Suppose $n_t^*(g,j) < n_t^*(g,k)$ where $j < k < i_t^*(g)$.

By using the proof of Theorem 3.2, it implies that

$$\Delta_t(g,n,k;X) \geq \Delta_t(g,n,j;X) \text{ for all n}$$

and

$$\Delta_t(g,n_t^*(g,k),k;X) \geq \Delta_t(g,n_t^*(g,j),k;X) \text{ for all k.}$$

Moreover, by the definition of $n_t^*(g,i)$, we know that

$$\Delta_t(g,n_t^*(g,k),k;X) \geq 0 \text{ and } \Delta_t(g,n_t^*(g,j),k;X) < 0 \text{ for all } j < k$$

which together imply

$$\begin{aligned}\Delta_t(g,n_t^*(g,k),k;X) &\geq \Delta_t(g,n_t^*(g,j),k;X)\\ &\geq \Delta_t(g,n_t^*(g,j),j;X) > 0\end{aligned}$$

It is clear that $\Delta_t(g,n_t^*(g,j),k;X) > 0$ contradicts $\Delta_t(g,n_t^*(g,j),k;X) > 0$, and the fact that $n_t^*(g,i)$ is nonincreasing in i for all $i < i_t^*(g)$.

Proof of (b): The proof is thoroughly analogous to that of part (a). □

For convenience sake, the structure of the optimal replacement policy derived by the above Theorems 3.2, 3.3 and Corollary 3.1 can be depicted in Figures 1 and 2, respectively:

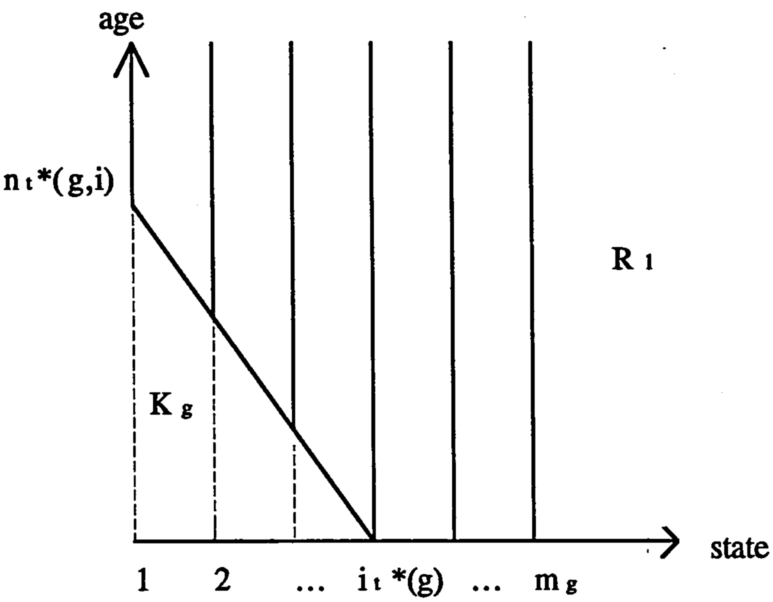

Figure 1. Optimal two-regions policy from Theorem 3.2 and Corollary 3.1(a)

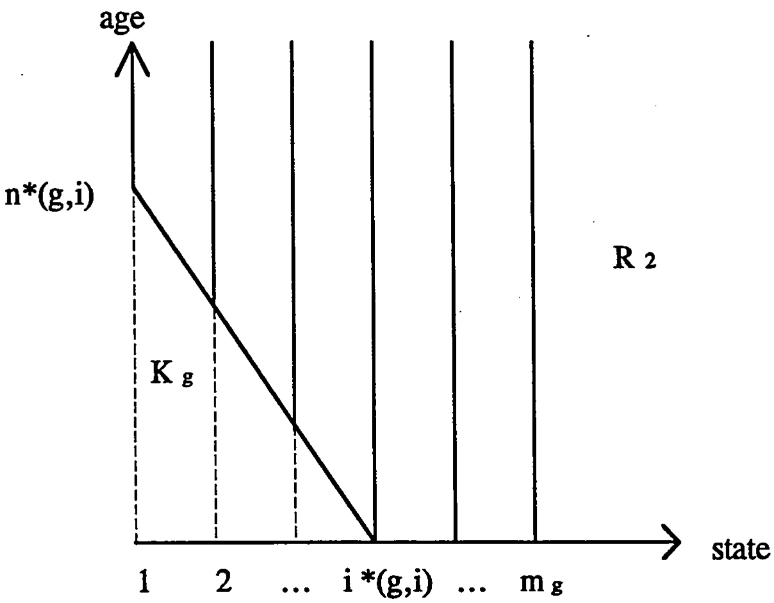

Figure2. Optimal two-regions policy from Theorem 3.3and Corollary 3.1(b)

Lemma 3.6: If α_t is nondecreasing in t and $\pi_t(g,n,i;X)=R_1$ for some t, then $f_t(g,n,i;X)$ is nondecreasing in t.

Proof:

(i) By Theorem 3.1, we have $\alpha_t = \alpha$ for all t ≥ t* where $\alpha \in [0,1]$. Hence,

$f_s(g,n,i;X) = f_t(g,n,i;X) = \overline{f}(g,n,i;X)$ for any s, t ≥ t*.

(ii) Let $s = t^*\text{-}1$

Since Y ⊃ X, it implies that $f(g,n,i;Y) \ge f_t(g,n,i;Y)$ for all t, we know

$$
\begin{aligned}
f_s(g,n,i;X) &= -c_1 + f_s(1,0,1;X) \\
&= -c_1 + r(1,0,1) + \beta\ [(1-\alpha_s)[\sum_{j\ge1} p_{10}(i,j)\overline{f}(1,1,j;X) \\
&\quad + \alpha_s \sum_{j\ge1} p_{10}(1,j) f(1,1,j;Y)] \\
&\le -c_1 + r(1,0,1) + \beta\ [(1-\alpha)[\sum_{j\ge1} p_{10}(1,j)\overline{f}(1,1,j;X) \\
&\quad + \alpha \sum_{j\ge1} p_{10}(1,j) f(1,1,j;Y)] \\
&= \overline{f}(g,n,i;X) \\
&= f_{s+1}(g,n,i;X)
\end{aligned}
$$

Suppose k < t*-1 and $f_k(g,n,i;X) \le f_{k+1}(g,n,i;X)$ holds. Then

$$
\begin{aligned}
f_{k-1}(g,n,i;X) &= -c_1 + f_{k-1}(1,0,1;X) \\
&= -c_1 + r(1,0,1) + \beta\ [(1-\alpha_{k-1})[\sum_{j\ge1} p_{10}(1,j)\overline{f_k}(1,1,j;X) \\
&\quad + \alpha_{k-1} \sum_{j\ge1} p_{10}(1,j) f(1,1,j;Y)]
\end{aligned}
$$

$$\leq -c_1 + r(1,0,1) + \beta\ [(1-\alpha_{k-1})[\sum_{j\geq 1} p_{10}(1,j)\overline{f}_{k+1}(1,1,j;X)$$
$$+ \alpha_{k-1}\sum_{j\geq 1} p_{10}(1,j) f(1,1,j;Y)]$$

$$\leq -c_1 + r(1,0,1) + \beta\ [(1-\alpha_k)[\sum_{j\geq 1} p_{10}(1,j)\overline{f}_{k+1}(1,1,j;X)$$
$$+ \alpha_k \sum_{j\geq 1} p_{10}(1,j) f(1,1,j;Y)]$$

$$= f_k(g,n,i;X)$$

From cases (i) and (ii), the proof is derived by induction. □

Lemma 3.7: If α_t is nondecreasing in t and $\pi_t(g,n,i;X) = k_g$ for some t, then $f_t(g,n,i;X)$ is nondecreasing in t.
Proof: The proof is similar to that of the above and is omitted. □

From the above lemmas, we derive the following theorem, implying that the conditional probability of technology breakthrough is nondecreasing in time and that the optimal value function is also nondecreasing. It means that we prefer prompt technology break-through which brings forth the increasing reward.

Theorem 3.4: If α_t is nondecreasing in t, then $f_t(g,n,i;X)$ is nondecreasing.
Proof: By Lemmas 3.6 and 3.7, we can derive the immediate result. □

Recall Theorem 3.1 and recursion (2), $f_t(g,n,i;X) = f_{t+1}(g,n,i;X)$ and $\Delta_t(g,n,i;X) = \Delta_{t+1}(g,n,i;X)$ for all $t \geq t^*$, it is sure that we have the optimal stationary policy over infinite horizon for all $t \geq t^*$. Obviously, this property is used to get the optimal initial decision without using the boundary conditions. Meanwhile, we apply the following forecast horizon theorem to obtain the same result. The algorithm for finding a forecast horizon is seen in Hopp (1987) and Nair (1989).

Theorem 3.5: If $\Delta_t(g,n,i;X)$ is non-zero for all t, then there exists a T* such that $\pi_t^T(g,n,i;X)=\pi_t^{T^*}(g,n,i;X)$ for all T ≥ T* and hence T = T * is a forecast horizon.

Proof: Since $f_t^T(g,n,i;X)=f_t(g,n,i;X)$ as $T\to\infty$; for any $\varepsilon > 0$, it is possible to choose T* sufficiently large to make

$$|\Delta_t^T(g,n,i;X)-\Delta_t(g,n,i;X)|\le\varepsilon \text{ for all } T\ge T^*.$$

Hence, $\Delta_t^T(g,n,i;X)$ will have the same sign as $\Delta_t(g,n,i;X)$, i.e., if $\Delta_t(g,n,i;X)>0$, then $\pi_t^T(g,n,i;X)=R_1$ and if $\Delta_t(g,n,i;X)<0$, then $\pi_t^T(g,n,i;X)=k_g$ for all T ≥ T*. The theorem is established. □

VI. CONCLUSIONS AND FURTHER STUDY

In this study, we have modeled the Markovian deterioration system with new knowledge impact resulting in a single technological improvement for the infinite horizon. We then derived the conditional probability distribution of technology breakthrough by using the concept in shock model (Aven and Gaarder (1987)). Under reasonable conditions, we have shown that the conditional probability of technology breakthrough remains constant after technology forecast period. This leads to the stationary optimal policy which has a special control limit structure with two properties: when a system has undergone beyond the control limit state, we have to undergo replacement regardless of equipment age; when a system is below control limit state, the control limit age is nonincreasing with respective to the deterioration state. It means that for the maximal total expected discounted reward under technological changes, the firms have to perform the equipment replacement following the control limit state and age as a threshold for economical reasons.

We also show that the acceleration of technology breakthrough will result in the increase of the optimal value function. Finally, we demonstrate that there exists a minimal forecast horizon such that the initial action is optimal, regardless of forecasts in later periods. It means that the forecast horizons are significant not only because they limit the computation and forecasting essential to arrive at an optimal decision, but also because they ensure that the optimal decision would be no different if relevant data for additional periods are forecasted.

In view of the management applications, this study points to the following further research issues:

- To develop efficient algorithms for forecast horizon (see Sethi and Chand (1979), Bean et al. (1985), Bes and Sethi (1988), Nair and Hopp (1992)).
- To study the case of partially observable Markov decision processes (see Monahan (1982), White (1991)).
- To generalize the equipment replacement problem with multiple technologies and switching costs (see Bylka et al. (1992)).
- To analyze the optimal policy sensitivity on equipment procurement cost, age, and conditional probability of technology breakthrough (see Hopp(1988), Wu and Hou (1992)).
- To apply our model on the business operation under dynamic environment where the innovation or adaptation decision is taken into account.

In addition, a future study can be extended on system acquisition where the make or buy decision is concerned.

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技術突破與馬可夫耗損下設備重置模式

吳森琪

摘 要

本文延伸 Nair(1989) 之技術性報廢與馬可夫耗損下設備重置模式，假設因科技新知識的衝擊累積，導致單一技術突破出現，促使較優性能之改良型設備上市。因此，是否續用現行設備或購置新型設備，以求取無限時間幅度下之最大總期望折現報酬，將成爲管理上之重要課題。有鑒於此，本研究之最佳政策，同時考量報酬函數與技術突破條件機率分佈函數。其次，設若報酬函數不僅爲設備狀態，亦爲使用年限之函數；有關技術突破條件機率分佈函數之建立與推證，則係沿修 Aven 與 Garrder(1987) 之衝擊模式而得。據此，引用隨機控制理論，建構最佳值函數之動態遞迴式。在合理假設條件下，證明技術預測時期存在，且自該時期以後，技術突破條件機率爲常數；而最佳重置政策結構，可以設備狀態與使用年限組成之特殊控制極限表之。再者，若技術突破機率爲時間之非遞減函數，則最佳值函數亦爲時間之非遞減函數。最後，證明最佳重置政策存在預測幅度。

關鍵詞：馬可夫耗損、技術突破、控制極限、最佳重置政策、預測幅度。