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Valuation of Spread and Basket Options

價差選擇權與一籃子選擇權之評價

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Abstract

This study adopts the unbounded-system distribution of the Johnson (1949) distribution family to approximate the basket/spread distribution and derive a versatile pricing model. This pricing model can price both basket and spread options, and thus, the risks of issuing both options can be consistently and efficiently integrated and managed. Furthermore, the pricing model can instantly price basket/spread options (almost as short in time as the Black-Scholes model (Black and Scholes, 1973)), and the results are quite accurate compared with the Monte Carlo simulation results. The method for computing Greeks is also presented. Finally, numerical examples are provided to demonstrate the implementation of the pricing model, and show the economic intuitions of Greeks for basket and spread options, and for an option portfolio consisting of both options.

【Keywords】basket options, spread options, martingale pricing method

摘 要

本研究採用 Johnson (1949) 分配族中的無邊界系統分配來近似一籃子 / 價差標的資產分配並推導出定價模型。該定價模型可以對一籃子選擇權和價差選擇權進行定價，因此可以一致且有效地整合及管理發行這兩種選擇權的風險。又，該定價模型可以即時對一籃子 / 價差選擇權進行定價（時間幾乎與 Black-Scholes (Black and Scholes, 1973) 評價模型一致），且與蒙地卡羅模擬的評價結果相比，顯示其定價結果相當準確。本研究還介紹了計算 Greeks 的方法。最後，數值範例展示定價模型的實作結果，並呈現一籃子選擇權、價差選擇權及兩種選擇權組成的選擇權組合等商品 Greeks 分別的經濟直覺。

【關鍵字】一籃子選擇權、價差選擇權、平賭過程評價方法

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1. Introduction

A spread option is a financial contract on the price difference between several assets. The underlying assets may include stocks, stock indexes, interest rates, foreign exchange rates, or commodities. A variety of spread options are widely traded both on exchanges and in over-the-counter markets. Investors may use them to speculate or hedge the spread (correlation) risk. For example, in the agriculture market, soybean crush options traded on the Chicago Board of Trade are written on the price difference between raw soybean and two soybean products — namely soybean oil and soybean meal. They can be used to lock in the producer's profit.

In the energy market, crack spread options written on the price spread between crude oil and several refined products are traded on the New York Mercantile Exchange. Electricity spark spread options traded in the over-the-counter market can be used to hedge the profit of producing electricity with natural gas. As for hedging interest-rate risks, interest-rate spread options, such as Constant Maturity Swap rate (CMS) spread options, can be used to hedge the spread between long- and short-term interest rates. Credit spread options can be traded on the credit spread between two counterparties with different credit-quality levels.[1]

After the collapse of the Bretton-Woods exchange rate system, the exchange rates between major currencies have become significantly volatile. Thus, managing currency risk becomes a vital issue, especially for multinational corporations involved in exporting and/or importing goods.[2] For example, a globally-diversified corporation may generate receivables in some currencies by exporting products and payables in other currencies by importing materials or equipment. To manage the multi-currency exchange rate risks of assets (receivables) and liabilities (payables), treasurers may use currency spread options to neutralize currency risks.

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1 For more information about the credit risk, the interest rate risk, and its related empirical studies, refer to Augustin, Sokolovski, Subrahmanyam, and Tomio (2022), Christensen, Kjær, and Veliyev (2023), Hasan, Marra, To, Wu, and Zhang (2023), Jaskowski and Rettl (2023), and Telg, Dubinova, and Lucas (2023).

2 Flood and Rose (1999) and Frömmel and Menkhoff (2003) indicate that the major floating exchange rates have become more and more volatile since 1973.

A basket option is another popular exotic option written on the value of a basket (portfolio) of assets, and is actively traded both on exchanges and in over-the-counter markets. Basket options are traded mainly for investing and hedging a portfolio of assets, including stocks, stock indexes, currencies, and commodities. For example, if an investor expects that there is a booming energy market in the near future, they can buy call options on a basket of energy products, such as crude oil, natural gas, and their refined products. Another example is that if a company has receivables in various currencies and worries about adverse fluctuations of exchange rates that may reduce their domestic-currency value, the treasurer may buy a basket put option on relevant currencies to hedge this risk.

The challenge of pricing and hedging basket/spread options mainly stems from the lack of an explicit distribution of the sum/difference of correlated lognormal variates, and thus, their closed-form pricing formulas and hedge ratios cannot be derived. Therefore, to price basket/spread options, a variety of numerical methods and approximate pricing formulas have been developed and extensively used in the marketplace. For basket options, Levy (1992) approximates the underlying basket by the lognormal distribution and matches the first two moments with the distribution of the underlying basket. Gentle (1993) approximates the underlying basket by a geometric average, which relies on the fact that the geometric average of the lognormal distribution is also lognormally distributed. Milevsky and Posner (1998) apply the reciprocal gamma distribution and Posner and Milevsky (1998) use the shifted lognormal distribution to derive the approximate pricing formula of the basket options. Ju (2002) applies the Taylor expansion method; Flamouris and Giamouridis (2007) use the Edgeworth expansion method, and Bae, Kang, and Kim (2011) extend Ju's approximation (Ju, 2002) to derive the approximate pricing formula of the basket options. Kan (2017) extends Levy (1992) by modifying the moment matching approach to develop a Black-Scholes-type (Black and Scholes, 1973) formula. Moreover, Rogers and Shi (1995), Carmona and Durrleman (2005), Xu and Zheng (2009), and Caldana, Fusai, Gnoatto, and Grasselli (2016) derive the lower and upper price bounds of the basket options. Bayer, Siebenmorgen, and Tempone (2018) and Choi (2018) focus on the numerical quadrature integration technique that can ease the curse of dimensionality, and numerical pricing results show that the pricing method is accurate and efficient.

For spread options, Shimko (1994) approximates the two asset spread option prices by the expansion method provided by Jarrow and Rudd (1982). Kirk (1995) extended the closed-form pricing formula for exchange options proposed by Margrabe in 1978 to derive an approximation pricing formula for two-asset spread options. Poitras (1998) uses the Bachelier approximation to approximate the price difference of the two assets directly as a normal variable and derives the pricing formula of spread options. Alexander and Scourse (2004) apply the bivariate normal mixture model to approximate the underlying spread and derive the approximate pricing formula. Li, Deng, and Zhou (2008) provide the price bounds for digital spread options and derive the approximate pricing formula of spread options using the quadratic approximation method. Hurd and Zhou (2010) introduce a new formula for spread options pricing based on Fourier analysis of the payoff function. Li, Zhou, and Deng (2010) provide a closed-form approximation method for pricing spread options using the extended Kirk's approximation method (Kirk, 1995). Wu and Chen (2009, 2011) apply the lognormal approximation technique for pricing the interest rate and Constant Maturity Swap spread options. Besides, a more detailed and informed survey of the research on the valuation of European basket and spread options is provided by Lyden (1996), Carmona and Durrleman (2003), and Lin, Chung, and Yeh (2016).

In recent years, the subprime-loan turmoil and European debt crisis have made financial markets more volatile and many financial institutions incur credit events, which makes hedging risks an even more vital issue.[3] Therefore, integrating the pricing and hedging models for various financial derivatives, and developing an efficient and consistent method for risk management have become important for academic and practitioner-oriented research. To integrate the pricing formulas of both basket and spread options, Borovkova, Permana, and van der Weide (2007) adopt the Lognormal-system (LS) distribution of the Johnson (1949) distribution family to approximate the (real) distribution of the basket/spread of assets, and then derive a versatile pricing formula which can

3 For example, Chou, Chen, and Yang (2003) study the valuation of covered warrant with credit risk, Yeh and Yu (2015) employ the SABR-LMM (LIBOR Market Model) model proposed by Mercurio and Morini (2009) to price interest rate derivatives, and Lin, Chuang, and Fang (2021) explore and analyze the valuation and risk management of rainfall index. In addition, Lin et al. (2016) review the existing literature for pricing and hedging derivatives. Lin, Chung, and Yeh (2017) review and summarize the empirical studies of derivatives markets by conducting a survey with more than 140 papers. All these studies reveal that risk management using derivatives has become an important task for academic researchers and market-practitioners.

be used to price and hedge both options.[4] For American basket and spread options, Borovkova, Permana, and van der Weide (2012) develop an integrated pricing method via a binomial tree model.

Numerical examinations show that the BPW model (Borovkova et al., 2007) can accurately and efficiently price both basket and spread options in most cases, but its accuracy decreases gradually with increasing volatilities, decreasing correlations among underlying assets, and increasing time to maturity. This phenomenon can be explained by the fact that the LS distribution has only three flexible parameters to fit the target distribution. Thus, the BPW model (Borovkova et al., 2007) cannot well capture the high-moment features of the target distribution and may cause some pricing error in some extreme situations.

This study aims to extend and improve the BPW model (Borovkova et al., 2007) by including the fourth parameter to approximate the target distribution. We adopt the unbounded-system (US) distribution of the Johnson (1949) distribution family to approximate the target distribution. The US distribution has four flexible parameters, which can help in better capturing the high-moment features of the target distribution even in the cases of high assets' volatilities, low correlations among underlying assets, and long time to maturity. Therefore, our resulting pricing formula can price both spread and basket options more accurately in these extreme cases. Besides improving model accuracy, the resulting pricing model is also derived in a closed form; thus, this model remains computational efficiency. Moreover, their Greeks can also be derived analytically, which helps market practitioners manage risks efficiently for both basket and spread options.

The rest of this article is organized as follows. Section 2 presents the market model, and introduces the Johnson (1949) distribution family and their relevant properties. The pricing formulae and their Greeks within the US-distribution framework are derived in Section 3. Section 4 provides some numerical studies to demonstrate the model implementation and examine the accuracy of the resulting pricing model. The conclusions are presented in the last section.

4 Chang, Chen, and Wu (2012) provide the analytical solution for the equation system of the moment-matching method presented in Borovkova et al. (2007).

2. The Model and Johnson Distribution Family

This section first presents the model setup, and then introduces the Johnson (1949) distribution family and shows how it can be used to approximate the unknown distribution of a basket/spread of underlying assets.

2.1 The Basket/Spread of Underlying Assets

Assume that trading takes place continuously over a time interval $[0,\mathcal{T}]$, $0<\mathcal{T}<\infty$. The uncertainty is described by a filtered probability space $(\Omega, F, Q, \{\mathcal{F}_t\}_{t\in[0,\mathcal{T}]})$, where the filtration is generated by the correlated standard Brownian motions denoted by $W_i(t), i=1,2,\ldots,N$, and their instantaneous correlations between $W_i(t)$ and $W_j(t), 1\le i\neq j\le N$, are denoted by $\rho_{i,j}$. The measure Q represents the risk-neutral probability measure.

Consider N underlying assets whose dynamics under the risk-neutral probability measure Q are assumed to be the following stochastic differential equations:

$$\frac{dS_i(t)}{S_i(t)}=\mu_i dt+\sigma_i dW_i(t), i=1,2,\ldots,N, \tag{1}$$

where μ_i and σ_i represent the drift and diffusion terms, respectively.[5] Their prices at time T conditional on time-0 information can be derived by using the Itô Lemma as follows:

$$S_i(T)=S_i(0)\exp\left\{\left(\mu_i-\frac{1}{2}\sigma_i^2\right)T+\sigma_i W_i(T)\right\}. \tag{2}$$

Therefore, within the model setting, the time-T price of the underlying asset follows a lognormal distribution.

The model, presented by equation (1), can be straightforwardly generalized by including other risk factors, such as stochastic interest rates (e.g., Kijima and Muromachi, 2001; Bernard, Le Courtois, and Quittard-Pinon, 2008; Wu and Chen, 2007a, 2007b), and

5 The model can be easily applied to a variety of underlying assets by adjusting the setting of the drift terms. For example, if S stands for the foreign exchange rate, then $\mu=r_d-r_f$, where r_d and r_f represent the domestic and foreign risk-free interest rates, respectively. If S denotes an equity index, then $\mu=r_d-q$, where q represents the dividend yield rate. If S represents a forward or futures price, then μ=0, which is the same with the model setting specified in Borovkova et al. (2007). In addition, σ_i can be replaced by a time-varing process.

price jumps (e.g., Merton, 1976; Metwally and Atiya, 2002; Flamouris and Giamouridis, 2007; Ross and Ghamami, 2010). Within the framework of these two extended models, the time-*T* price of the underlying asset remains a lognormal distribution, and thus, their pricing methods for basket/spread options are similar to those derived within the model setting given in equation (1). Our purpose is to examine the performance of the Johnson (1949) distribution family, which is used to approximate the distribution of a basket/spread of underlying assets (or simply lognormal variates). Hence, to focus on the purpose of this study, we confine our model setting of the underlying assets to a geometric Brownian motion presented by equation (1).

2.2 The Basket/Spread of Underlying Assets

Since both a basket or spread of underlying assets can be expressed in the same form, we integrate them hereafter as a generalized basket (*GB*) defined as follows:

$$GB(T) = \sum_{i=1}^{N} \alpha_i S_i(T)\,, T \in [0, \mathcal{T}], \tag{3}$$

where $\alpha_i \in R$ stands for the unit number of the *i*th asset. If $\forall \alpha_i \in R^+$, then the *GB* represents a basket of underlying assets; if $\exists \alpha_i < 0$, then the *GB* represents a spread. Though the exact distribution of the *GB* is unknown, its first four moments can be computed and are presented in the Preposition 1.

Proposition 1. The first four moments of the *GB*(*T*) are computed as follows:

$$\mathrm{E}^{Q}[GB(T)] = \sum_{i=1}^{N} S_i^*(0) e^{\mu_i T},$$

$$\mathrm{E}^{Q}[GB^2(T)] = \sum_{i=1}^{N} \sum_{j=1}^{N} S_i^*(0) S_j^*(0) e^{(\mu_i+\mu_i+\eta_{i,j})T},$$

$$\mathrm{E}^{Q}[GB^3(T)] = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} S_i^*(0) S_j^*(0) S_k^*(0) e^{(\mu_i+\mu_i+\mu_k+\eta_{i,j}+\eta_{i,k}+\eta_{j,k})T},$$

$$\mathrm{E}^{Q}[GB^4(T)] =$$
$$\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} S_i^*(0) S_j^*(0) S_k^*(0) S_l^*(0) e^{(\mu_i+\mu_i+\mu_k+\mu_l+\eta_{i,j}+\eta_{i,k}+\eta_{i,l}+\eta_{j,k}+\eta_{j,l}+\eta_{k,l})T},$$

where $S_i^*(0) = \alpha_i S_i(0)$, $\eta_{i,j} = \rho_{i,j}\sigma_i\sigma_j$, and other notations are defined accordingly.

Based on Proposition 1 and some statistical computation, the mean ($\mathcal{M}$), variance ($\mathcal{V}$), skewness ($\mathcal{SK}$), and kurtosis ($\mathcal{K}$) of the $GB(T)$ can be derived as follows:

$$\mathcal{M} = \mathrm{E}^{Q}[GB(T)], \tag{4}$$

$$\mathcal{V} = \mathrm{E}^{Q}[(GB(T) - \mathcal{M})^2], \tag{5}$$

$$\mathcal{SK} = \mathrm{E}^{Q}\left[\left(\frac{GB(T) - \mathcal{M}}{\sqrt{\mathcal{V}}}\right)^3\right], \tag{6}$$

$$\mathcal{K} = \mathrm{E}^{Q}\left[\left(\frac{GB(T) - \mathcal{M}}{\sqrt{\mathcal{V}}}\right)^4\right]. \tag{7}$$

These four characteristics can be exactly computed using present market data.

2.3 The Johnson Distribution Family

The Johnson (1949) distribution family is a collection of probability distributions, which are transformed from standard normal distributions via three types of functions with four parameters. Let Z stand for a standard normal distribution and X denote a Johnson distribution. The relation between Z and X is presented by:

$$X = a + b\phi\left(\frac{Z - c}{d}\right), \tag{8}$$

where a is a location parameter, b is a scale parameter, and c and d are shape parameters. The transformation of a standard normal distribution, denoted by ϕ, falls into three types: a lognormal system, an unbounded system, and a bounded system, which are specifically presented as follows:

$$\phi(x) = \begin{cases} e^x & \text{(lognormal system)}, \\ (e^x - e^{-x})/2 & \text{(unbounded system)}, \\ 1/(1 + e^{-x}) & \text{(bounded system)}. \end{cases}$$

The probability density functions of each system can be derived and presented as follows.

Definition 1. Let X denote the Johnson distribution, and a, b, c, and d are the four parameters given in equation (8). The probability density functions of each system in the Johnson distribution family are given as follows:

(a) Lognormal System (LS)

$$f_{\mathrm{LS}}(x) = \frac{bd}{\sqrt{2\pi}(x-a)} \exp\left\{-\frac{1}{2}\left[c + d\ln\left(\frac{x-a}{b}\right)\right]^2\right\}, \tag{9}$$

where $(x-a)/b > 0, -\infty < a < \infty, |b| = 1, -\infty < c < \infty,$ and $d > 0$.

(b) Unbounded System (US)

$$f_{\mathrm{US}}(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{\sqrt{(x-a)^2 + b^2}} \exp\left\{-\frac{1}{2}\left[c + d\sinh^{-1}\left(\frac{x-a}{b}\right)\right]^2\right\}, \tag{10}$$

where $-\infty < x < \infty, -\infty < a < \infty, b > 0, -\infty < c < \infty,$ and $d > 0$.

(c) Bounded System (BS)

$$f_{\mathrm{BS}}(x) = \frac{1}{\sqrt{2\pi}} \frac{b^2 d}{(x-a)(b-x+a)} \exp\left\{-\frac{1}{2}\left[c + d\ln\left(\frac{x-a}{b-x+a}\right)\right]^2\right., \tag{11}$$

where $a < x < a+b, -\infty < a < \infty, b > 0, -\infty < c < \infty,$ and $d > 0$.

The advantage of the Johnson distribution family lies in its rich pair of skewness and kurtosis. To express this feature more explicitly, we present Figure 1 with the vertical axis representing the kurtosis ($\mathcal{K}$) and horizontal axis representing the square of skewness ($\mathcal{SK}^2$), and its coordinate is denoted by $(\mathcal{SK}^2, \mathcal{K})$. Figure 1 represents all possible pairs of $\mathcal{SK}^2$ and $\mathcal{K}$.[6] For example, the standard normal distribution is known to have $\mathcal{SK}^2 = 0$ and $\mathcal{K} = 3$, which is located at (0,3) and displayed by a circle.

The pair $(\mathcal{SK}^2, \mathcal{K})$ of the f distribution is presented by a curve denoted by $Curve_{\mathrm{LS}}$. The upper area, denoted by $Area_{\mathrm{US}}$, describes the pair $(\mathcal{SK}^2, \mathcal{K})$ which can be obtained from the US distribution. The middle area, denoted by $Area_{\mathrm{BS}}$, shows the pair $(\mathcal{SK}^2, \mathcal{K})$ which can be obtained by the BS distribution. The bottom area, denoted by Impossible Area, depicts the pair $(\mathcal{SK}^2, \mathcal{K})$ which cannot be captured by the Johnson distribution family. Therefore, if the pair $(\mathcal{SK}^2, \mathcal{K})$ of the GB of underlying assets belongs to any one of the possible areas, then one of the Johnson distribution family can accurately approximate the target distribution by matching its first four moments.

6 Figure 1 is plotted based on the limiting properties of the skewness and kurtosis of the Johnson distribution family.

Most market data exhibit nonzero skewness and higher kurtosis. This also holds for the *GB*, especially in the cases of higher volatilities, lower correlations among the underlying assets, and longer time to maturity. As shown in Figure 1, the distribution located in the $Area_{\mathrm{US}}$ has relatively higher kurtosis than the distribution in the $Area_{\mathrm{BS}}$. Thus, the US distribution is more capable of approximating the *GB* distribution. Empirical examinations with market data show that $(\mathcal{SK}^2, \mathcal{K})$ located in the $Area_{\mathrm{BS}}$ may only occur in a very extreme and unreal situation.[7] Therefore, this study does not adopt the BS distribution to fit the *GB* distribution.

Since the US distribution has one more flexible parameter than the LS distribution, $Curve_{\mathrm{LS}}$ lies on the edge of $Area_{\mathrm{US}}$. Therefore, the US distribution is much more versatile and can fit the *GB* distribution better than the LS distribution. Nonetheless, Borovkova et al. (2007) adopt the LS distribution to approximate the *GB* distribution; consequently, their resulting model has limited capacity to capture a variety of real skewness and kurtosis. The aforementioned mismatch with the real skewness and kurtosis may cause some pricing error, especially in the cases of higher volatilities, lower correlations among underlying assets, and longer time to maturity. This phenomenon is illustrated by the examples presented in Figures 2 and 3, which show that the US distribution can fit the *GB* distribution better than the LS distribution. In addition, this study matches the first four moments of the four-parameter US distribution with the *GB* distribution. Thus, the US distribution can approximate the *GB* distribution.[8] In summary, to enhance the pricing accuracy and retain computational efficiency, this study adopts the US distribution to approximate the *GB* distribution.

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7 Thanks to the anonymous reviewers for the suggestions about the empirical examination of the BS distribution. Appendix A provides the pair of $(\mathcal{SK}^2, \mathcal{K})$ of the GB distribution based on the numerical examinations from Tables 3 to 8.

8 Thanks to the anonymous reviewers for the suggestions about the theoretical foundation for the US distribution as an approximate distribution for the *GB* distribution. Based on the theoretical foundation of the Edgeworth series expansion method, matching the second or higher-order moments of both the underlying and approximating distributions shows that the underlying distribution can be approximated by the approximating distribution in terms of an Edgeworth series expansion. For more information, refer to Jarrow and Rudd (1982).

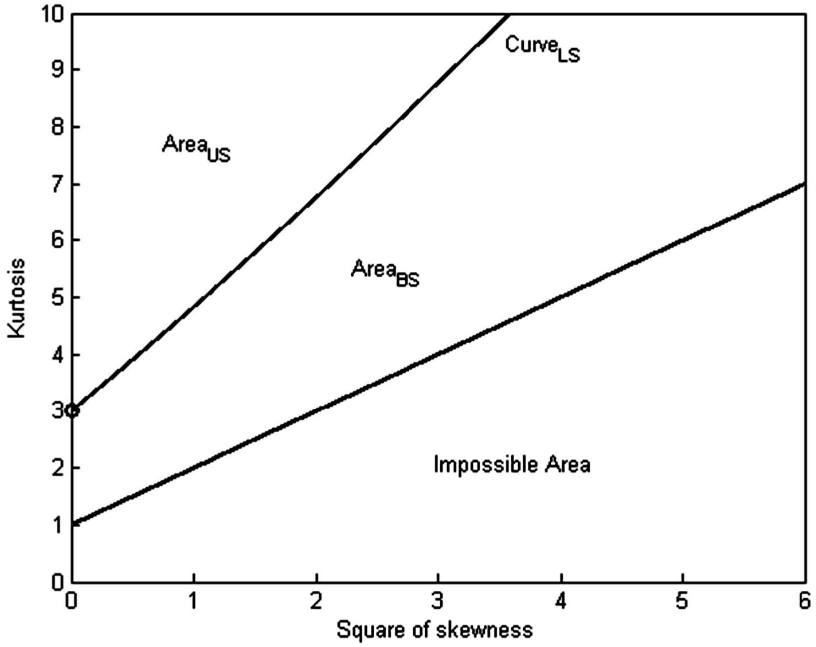

Figure 1 Pairs of $(\mathcal{SK}^2, \mathcal{K})$

Note: Figure 1 depicts the pair $(\mathcal{SK}^2, \mathcal{K})$ that can be yielded by the US, LS, and BS distributions, respectively.

We present the mean, variance, skewness, and kurtosis of the US distribution in the following proposition and their derivations in Appendix B.

Proposition 2. The four characteristics of the US distribution are presented as follows:

$$\mathcal{M}_{\mathrm{US}}(a,b,c,d) = a - b\omega^{\frac{1}{2}}\sinh(\Omega), \tag{12}$$

$$\mathcal{V}_{\mathrm{US}}(a,b,c,d) = \frac{b^2}{2}(\omega-1)(\omega\cosh(2\Omega)+1), \tag{13}$$

$$\mathcal{SK}_{\mathrm{US}}(a,b,c,d) = \frac{-\sqrt{\omega(\omega-1)}[\omega(\omega+2)\sinh(3\Omega)+3\sinh(\Omega)]}{\sqrt{2[\omega\cosh(2\Omega)+1]^3}}, \tag{14}$$

$$\mathcal{K}_{\mathrm{US}}(a,b,c,d) = \frac{\omega^2(\omega^4+2\omega^3+3\omega^2-3)\cosh(4\Omega)+4\omega^2(\omega+2)\cosh(2\Omega)+3(2\omega+1)}{2[\omega\cosh(2\Omega)+1]^2}, \tag{15}$$

where $\Omega = c/d$ and $\omega = exp(1/d^2)$.

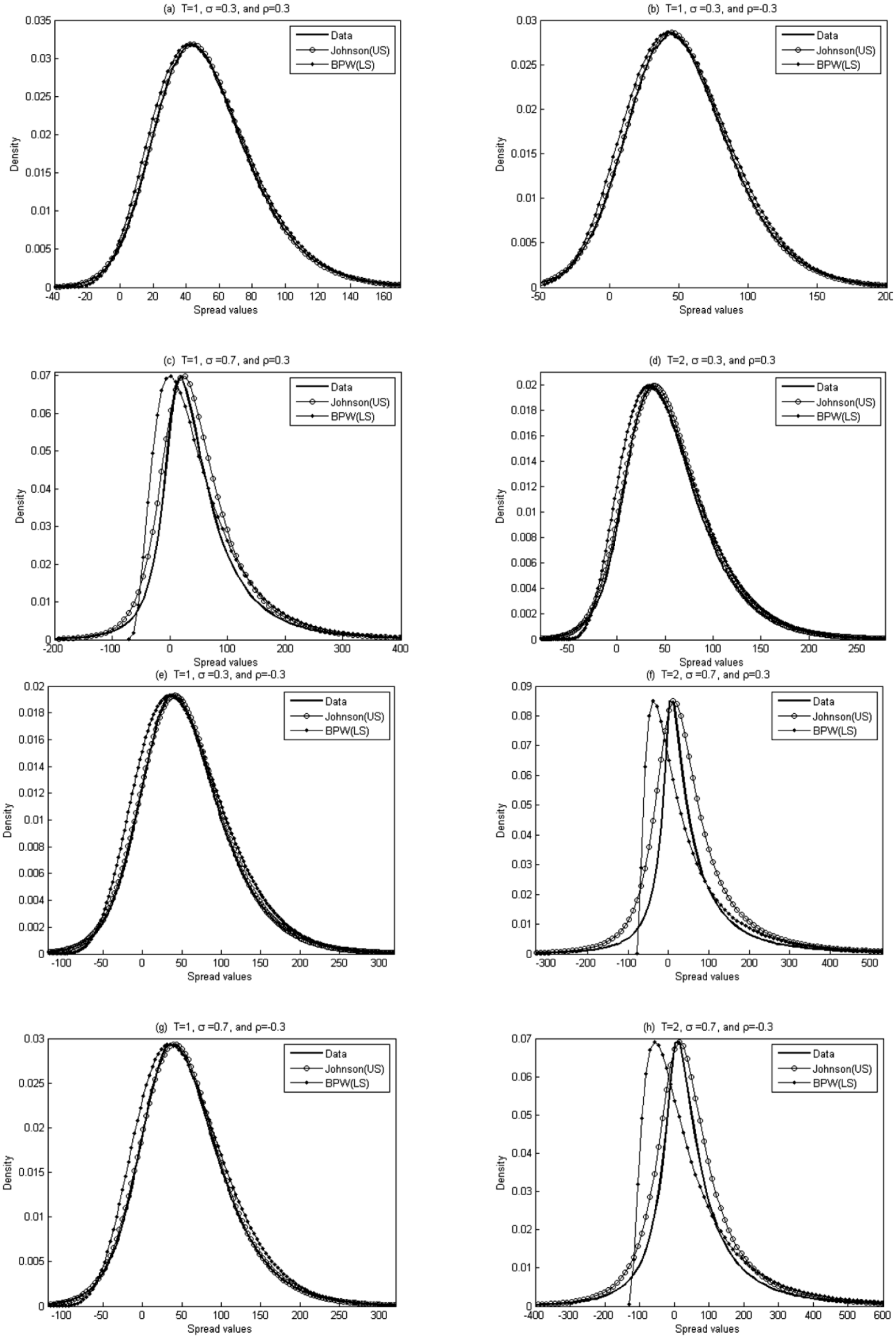

Figure 2 Comparison between the US and LS Systems

Note: Figure 2 provides the real distribution and two approximate distributions of $GB(T) = S_1(T) - S_2(T)$ with $S_1(0) = S_2(0) = 100$. The volatilities (assumed by $\sigma_1 = \sigma_2 = \sigma$), correlation, and time to maturity are presented in each plot.

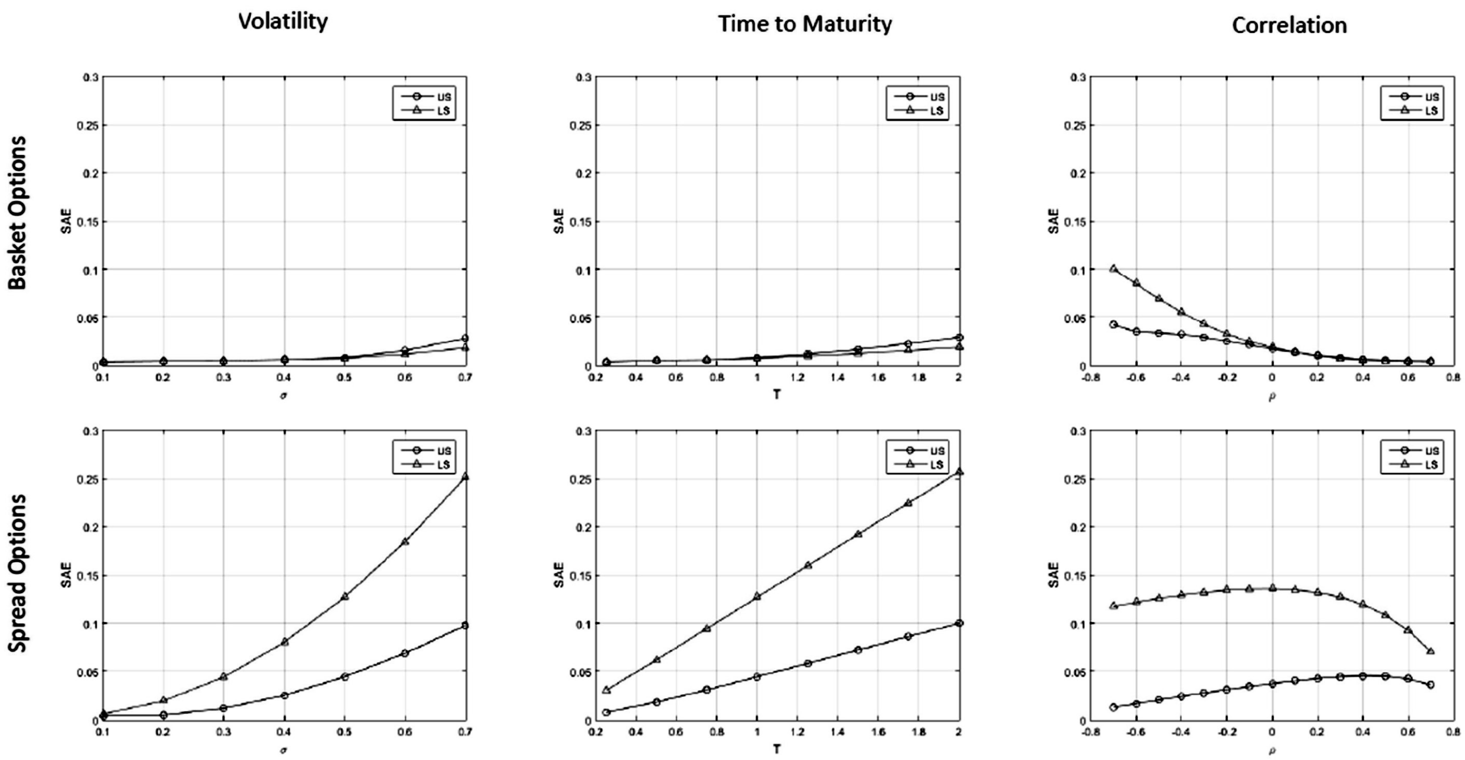

Figure 3 The Sum of Absolute Errors of Distribution of the US and LS

Note: Figure 3 provides the sum of absolute errors (*SAE*) of distribution of the US and LS. All results are based on the following parameters. The base case includes: the initial stock price $S_1(0) = 100$ and $S_2(0) = 50$, the volatilities $\sigma_1 = \sigma_2 = 50\%$, the correlation between two assets is 30%, the time to maturity is 1 year; the risk-free rate is 5%, the dividend yield is 0%, and the strike price is 150 for the basket options and 50 for the spread options. The first row means the *SAE* of $GB(T) = S_1(T) + S_2(T)$ and the second row means the *SAE* of $GB(T) = S_1(T) - S_2(T)$. Other things being equal, the first column shows the volatility case: increasing the volatilities from 10% to 70% with 10% increment. The second column shows the time to maturity case: increasing the time to maturity from 0.25 year to 2 year with 0.25 year increment. The third column shows the correlation case: increasing the correlation from -70% to 70% with 10% increment.

3. Pricing Formula of the GB Options with the US Distribution

This section first presents the procedure to find a matching US distribution to approximate the *GB* distribution using the moment-matching method, then derives the pricing formula for the *GB* options, and finally, show the computation of hedging Greeks.

3.1. The Moment-Matching Method for the US Distribution

As noted above, the challenge of pricing *GB* options mainly stems from the lack of an exact distribution of the *GB*; as a result, their pricing formulas can not be derived in precisely. To improve the BPW model (Borovkova et al., 2007), we adopt the US distribution family with the four correct characteristics presented in equations (4)-(7) to approximate the *GB* distribution. To choose a matching US distribution to approximate the *GB* distribution, we equalize the first four characteristics of the US distribution to those of the *GB* distribution, and obtain the following equation system:

$$\begin{aligned} \mathcal{M}_{\mathrm{US}}(a,b,c,d) &= \mathcal{M} \\ \mathcal{V}_{\mathrm{US}}(a,b,c,d) &= \mathcal{V} \\ \mathcal{SK}_{\mathrm{US}}(a,b,c,d) &= \mathcal{SK} \\ \mathcal{K}_{\mathrm{US}}(a,b,c,d) &= \mathcal{K}\,. \end{aligned} \tag{16}$$

By solving this equation system and denoting the solution by $(\bar{a}, \bar{b}, \bar{c}, \bar{d})$, we can determine a matching US distribution to approximate the *GB* distribution.[9]

Because equation (16) is a nonlinear equation system, solving this equation is not straightforward and must resort to a numerical method. Tuenter (2001) proposes a root-finding algorithm built on the Newton-Raphson method and shows the sufficient conditions for convergence. Therefore, we adopt the method proposed by Tuenter (2001) to solve equation (16), and arrange and reduce their results into the following three steps.

Step 1: Compute the initial value ω_0 as follows:

$$\omega_0 = \sqrt{\sqrt{2\mathcal{K} - 2} - 1}.$$

9 The moment-matching approach is also used for the pricing of Asian options, such as in Chang and Tsao (2011) and Lo, Palmer, and Yu (2014), and guaranteed minimum withdrawal benefits, such as in Milevsky and Salisbury (2006) and Yang, Wang, and Liu (2020).

Step 2: Set a tolerable error ε. If $|\omega_i - \omega_{i-1}| < \varepsilon$, then $\bar{\omega} = \omega_i$. Otherwise, continue the following iteration:

$$\omega_i = \omega_{i-1} - \frac{f(\omega_{i-1}) - \mathcal{SK}^2}{f'(\omega_{i-1})}$$

$$= \omega_{i-1} + \frac{2}{3}\frac{(\omega_{i-1}-1-m)(\omega_{i-1}+2+m/2)^2 - \mathcal{SK}^2}{(\omega_{i-1}+2+m/2)\left[m+2(\mathcal{K}+3)(\omega_{i-1}+1)/\left(\omega_{i-1}^2+2\omega_{i-1}+3\right)^2\right]},$$

where $i = 1, 2, \ldots$, and $m = -2 + \sqrt{4 + 2\left(\omega_{i-1}^2 - \frac{\mathcal{K}+3}{\omega_{i-1}^2+2\omega_{i-1}+3}\right)}$.

Step 3: With $\bar{\omega}$ computed in step 2, we can compute $\bar{\Omega}$, $\bar{m}$, and the four parameters $\left(\bar{a}, \bar{b}, \bar{c}, \bar{d}\right)$ as follows:

$$\bar{a} = \mathcal{M} + \bar{b}\sqrt{\bar{\omega}}\sinh(\bar{\Omega}), \tag{17}$$

$$\bar{b} = \frac{\sqrt{\mathcal{V}}}{(\bar{\omega}-1)\sqrt{\frac{\bar{\omega}+1}{2\bar{m}}}}, \tag{18}$$

$$\bar{c} = \bar{d}\bar{\Omega}, \tag{19}$$

$$\bar{d} = \frac{1}{\sqrt{ln(\bar{\omega})}}, \tag{20}$$

where

$$\bar{\Omega} = -\mathrm{sign}(\mathcal{SK})\sinh^{-1}\left(\sqrt{\frac{\bar{\omega}+1}{2\bar{\omega}}\left(\frac{\bar{\omega}-1}{\bar{m}} - 1\right)}\right), \tag{21}$$

$$\bar{m} = -2 + \sqrt{4 + 2\left(\bar{\omega}^2 - \frac{\mathcal{K}+3}{\bar{\omega}^2 + 2\bar{\omega} + 3}\right)}, \tag{22}$$

and $\mathcal{M}$, $\mathcal{V}$, $\mathcal{SK}$, and $\mathcal{K}$ can be computed by equations (4)-(7) based on current market data.

The Newton-Raphson method can compute $\left(\bar{a}, \bar{b}, \bar{c}, \bar{d}\right)$ in a fraction of a second, and then determine the matching US distribution to approximate the *GB* distribution. For the numerical examples presented in Section 4, the Newton-Raphson method converges within five iterations by taking approximately 2×10^{-5} seconds. Thus, it ensures that the resulting pricing formulas can be instantly computed.

3.2. Pricing Formula of the Basket/Spread Options with the US Distribution

Basket/Spread options are financial contracts on the basket/spread of multiple underlying assets whose final payoffs can be jointly defined as follows:

$$GBC(T) = \mathrm{Max}(GB(T) - K, 0), \tag{23}$$

$$GBP(T) = \mathrm{Max}(K - GB(T), 0), \tag{24}$$

where K represents the strike price, and *GBC* and *GBP* denote the call and put options on the *GB*, respectively. The generalized basket (*GB*) is defined as follows:

$$GB(T) = \sum_{i=1}^{N} \alpha_i S_i(T)\,, T \in [0, \mathcal{T}],$$

where $\alpha_i \in R$ represents the unit number of the *i*th asset. If $\forall \alpha_i \in R^+$, then the *GB* represents a basket of underlying assets; if $\exists \alpha_i < 0$, then the *GB* represents a spread.

Based on the martingale pricing method, the pricing formulas of the *GB* options can be derived by computing the following expectations:

$$GBC(T) = e^{-rT} E^{\mathcal{Q}}[\mathrm{Max}(GB(T) - K, 0)], \tag{25}$$

$$GBP(T) = e^{-rT} E^{\mathcal{Q}}[\mathrm{Max}(K - GB(T), 0)]\,. \tag{26}$$

However, as mentioned above, the distribution of the *GB*(*T*) is unknown, resulting in the above expectations cannot be analytically derived. Instead, the US distribution is employed to approximate the *GB* distribution and then to derive the approximate pricing formula of the *GB* option. Once the matching US distribution is obtained following the procedure outlined in section 3.1, the approximate pricing formulas of the *GB* options can be derived and presented as follows. The derivation is presented in Appendix C.

Theorem 1. The pricing formulae of the *GB* call and put options are as follows:

$$\begin{aligned} GBC(T) = e^{-rT} \Bigg[& \mathcal{M} - K + (K - \bar{a})\Phi(R) + \frac{\bar{b}}{2} exp\left(\frac{1 + 2\bar{c}\bar{d}}{2\bar{d}^2}\right)\Phi\left(R + \frac{1}{\bar{d}}\right) \\ & - \frac{\bar{b}}{2} exp\left(\frac{1 - 2\bar{c}\bar{d}}{2\bar{d}^2}\right)\Phi\left(R - \frac{1}{\bar{d}}\right)\Bigg], \end{aligned} \tag{27}$$

$$\begin{aligned} GBP(T) = e^{-rT} \Bigg[& (K - \bar{a})\Phi(R) + \frac{\bar{b}}{2} exp\left(\frac{1 + 2\bar{c}\bar{d}}{2\bar{d}^2}\right)\Phi\left(R + \frac{1}{\bar{d}}\right) \\ & - \frac{\bar{b}}{2} exp\left(\frac{1 - 2\bar{c}\bar{d}}{2\bar{d}^2}\right)\Phi\left(R - \frac{1}{\bar{d}}\right)\Bigg], \end{aligned} \tag{28}$$

where $\mathcal{M}$ is defined in equation (4), $R = \bar{c} + \bar{d}\sinh^{-1}\left(\frac{K-\bar{a}}{\bar{b}}\right)$, $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}z^2} dz$, and $\bar{a}, \bar{b}, \bar{c}$, and $\bar{d}$ are given in equations (17)-(20).

With inheriting from the merits of the BPW model (Borovkova et al., 2007), the derived pricing models given in equations (27) and (28) can together price both basket and spread options, and thus, the pricing and hedging of the two options can be managed consistently and efficiently. Furthermore, the pricing models improve the pricing capacity of the BPW model (Borovkova et al., 2007) by incorporating one more flexible parameter, which can capture the features of the first four moments of the *GB* distribution. Therefore, the resulting pricing models can significantly reduce the pricing error, especially in the situations of higher asset volatilities, lower correlations among underlying asset prices, and a longer time to maturity.

3.3 Hedging Ratio

Hedging the *GB* options is as important as pricing them for investment banks. Therefore, this subsection examines how to compute the hedging ratios (or the Greeks) of the *GB* options. Note that though the pricing formulas given in equations (27) and (28) are presented in a close form, their Greeks cannot be analytically derived because $\bar{a}, \bar{b}, \bar{c}$, and $\bar{d}$ must be computed via the Newton-Raphson method. To overcome this obstacle, this subsection suggests that the end-users should compute the Greeks directly by their definitions. For demonstration, the definitions of Greeks are presented as follows.

Definition 2. The Greeks of the *GB* options can be approximately computed by the following formulas:

$$
\begin{aligned}
\Delta_i^{GBC} &= \frac{GBC(S_i(0)+\delta) - GBC(S_i(0))}{\delta}, \\
\Gamma_i^{GBC} &= \frac{\Delta_i^{GBC}(S_i(0)+\delta) - \Delta_i^{GBC}(S_i(0))}{\delta}, \\
\nu_i^{GBC} &= \frac{GBC(\sigma_i+\delta) - GBC(\sigma_i)}{\delta}, \\
\psi_{i,j}^{GBC} &= \frac{GBC(\rho_{i,j}+\delta) - GBC(\rho_{i,j})}{\delta}, \\
\xi^{GBC} &= \frac{GBC(r+\delta) - GBC(r)}{\delta}, \\
\theta^{GBC} &= \frac{GBC(T+\delta) - GBC(T)}{\delta},
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_i^{GBP} &= \frac{GBP(S_i(0)+\delta) - GBP(S_i(0))}{\delta}, \\
\Gamma_i^{GBP} &= \frac{\Delta_i^{GBP}(S_i(0)+\delta) - \Delta_i^{GBP}(S_i(0))}{\delta}, \\
\nu_i^{GBP} &= \frac{GBP(\sigma_i+\delta) - GBP(\sigma_i)}{\delta}, \\
\psi_{i,j}^{GBP} &= \frac{GBP(\rho_{i,j}+\delta) - GBP(\rho_{i,j})}{\delta}, \\
\xi^{GBP} &= \frac{GBP(r+\delta) - GBP(r)}{\delta}, \\
\theta^{GBP} &= \frac{GBP(T+\delta) - GBP(T)}{\delta},
\end{aligned}
$$

where δ is a sufficiently small number and the other parameters are fixed as a constant in the computation of each Greek.

Based on the pricing formulas presented in Theorem 1, the approximate Greeks of the *GB* options can be instantly and accurately computed via the above Greeks computation method.[10] The accuracy depends on the size of δ we choose; that is, the smaller the size of δ, the more accurate the computed Greeks. Note that the size of chosen δ will not affect the computation time; accordingly, the above Greek formulas can also be viewed as close-form formulas. As a rule of thumb, we may set $\delta = 10^{-5}$ (or even smaller) for each case, which can uniformly yield sufficiently accurate Greeks.

4. Numerical Studies

This section provides some numerical examples to examine the accuracy of the resulting pricing models and then presents some sensitivity analysis for the Greeks.

10 It is not unreasonable to view the computation of Greeks as a (quasi-) closed-form model since their solutions generally converges within five iterations with the Newton-Raphson method. For computing each option value, the presented pricing formula takes approximately 2.33×10^{-4} of a second, which is almost the same as the 1.6×10^{-4} taken by the Black and Scholes (1973) formula.

4.1 Numerical Examinations

Borovkova et al. (2007) adopt the LS distribution of the Johnson distribution family to derive a versatile pricing formula which can accurately and efficiently price both the basket options and spread options. However, our numerical examination below reveals that the BPW model (Borovkova et al., 2007) yields relatively higher pricing errors in cases of higher asset volatilities, lower correlations among underlying assets, and longer time to maturity. To improve the pricing capability, this study adopts the US distribution of the Johnson distribution family to approximate the *GB* distribution.

To examine the accuracy of our model, we first employ the numerical examples provided in Borovkova et al. (2007) and compare the results computed via the BPW model (Borovkova et al., 2007) and our pricing model. Table 1 presents the market scenarios provided in Borovkova et al. (2007), and the pricing results are given in Table 2. Clearly, our model yields almost the same prices as those computed from the Monte Carlo simulation, while the BPW method (Borovkova et al., 2007) shows slight deviations from the Monte Carlo simulation.

Note that the six market scenarios provided in Borovkova et al. (2007) are composed of low volatilities and high correlations among the underlying assets, and short time to maturity. Under these conditions, the BPW approximate pricing formulas (Borovkova et al., 2007) easily perform well. However, our pricing formulas can accurately price the *GB* options even in difficult situations, such as high volatilities and low correlations among the underlying assets, and longer time to maturity. To support our claim, we provide more comprehensive numerical examples and show that our model can deal with these difficult situations better than the BPW model (Borovkova et al., 2007). The results are presented in Tables 3, 4, 5, 6, 7, and 8.

Lo et al. (2014) adopt a shifted reciprocal gamma distribution to approximate the distribution of the sum of lognormal variates. Therefore, this study also uses the same approximation method to derive the pricing formulas of the general basket options and their pricing results are also presented in Tables 3, 4, 5, 6, 7, and 8.

To evaluate the performance of each model by comparing it with the result computed based on the Monte Carlo simulation method, we provide the percentage pricing error (*PPE*), root of mean squared error (*RMSE*), and maximum absolute error (*MAE*), which are computed as follows:

$$PPE_{i,j} = \frac{P_{i,j} - P_{i,\mathrm{MC}}}{P_{i,\mathrm{MC}}},$$

$$RMSE_{i,j} = \sqrt{\frac{\sum_{i=1}^{N}\left(P_{i,j} - P_{i,\mathrm{MC}}\right)^2}{N}},$$

$$PPE_{i,j} = \max_{i\in\{1,2,\dots,N\}}\left|P_{i,j} - P_{i,\mathrm{MC}}\right|,$$

where $P_{i,j}$ means the ith price of model j and $j \in \{\mathrm{USD}, \mathrm{SLN}, \mathrm{SRG}\}$ $P_{i,\mathrm{MC}}$ means the ith price of the Monte Carlo method. For ease of reading, the *PPE* of each case greater than 10% will be marked by ***; between 5% and 10% by **; and between 1% and 5% by *. No asterisk means that the *PPE* is lower than 1%.

As shown clearly in Tables 3, 4, 5, 6, 7, and 8, our pricing model (denoted by USD) produces the prices almost identical to those computed with the Monte Carlo simulation even in difficult situations. In contrast, the SLN and SRG models produce prices close to those computed with Monte Carlo simulation in normal cases; however, their performance deteriorates significantly in difficult situations. Therefore, the numerical examination indicates that our pricing model can more robustly and accurately price both spread and basket options than the SLN and SRG models.

Regarding the computation efficiency, the resulting pricing formulas can price basket and spread options in a very small fraction of a second even though the parameters of the formulas should be computed via the Newton-Raphson method. For each option presented in Tables 3, 4, 5, 6, 7, and 8, the Newton-Raphson method converges within five iterations, taking approximately 2×10^{-5} of a second. In addition, the computation time of our pricing formulas for each option is approximately 2.33×10^{-4}, which is almost the same as 1.6×10^{-4} taken by the BS formula (Black and Scholes, 1973). This shows that our pricing model can instantly price basket and spread options, and thus, it justifies the use of Definition 2 to compute the Greeks via their definitions.

4.2. Numerical Examples with Market Data

In this section, we present some numerical examples using market data and illustrate how to estimate the parameter. To make the pricing results more readable and comparable, we select three representative companies from three different industries: Taiwan Semiconductor Manufacturing Co., Ltd. (2330), Evergreen Marine Corporation (2603),

Table 1 The *GB* Parameters of Each Numerical Example Provided in Borovkova et al. (2007)

Terms	*GB1*	*GB2*	*GB3*	*GB4*	*GB5*	*GB6*
$S_i(0)$	[100, 120]	[150, 100]	[110, 90]	[200, 50]	[95, 90, 105]	[100, 90, 95]
σ_i	[0.2, 0.3]	[0.3, 0.2]	[0.3, 0.2]	[0.1, 0.15]	[0.2, 0.3, 0.25]	[0.25, 0.3, 0.2]
α_i	[-1, 1]	[-1, 1]	[0.7, 0.3]	[-1, 1]	[1, -0.8, -0.5]	[0.6, 0.8, -1]
$\rho_{i,j}$	$\rho_{1,2} = 0.9$	$\rho_{1,2} = 0.3$	$\rho_{1,2} = 0.9$	$\rho_{1,2} = 0.8$	$\rho_{1,2} = \rho_{2,3} = 0.8$ $\rho_{1,3} = 0.8$	$\rho_{1,2} = \rho_{2,3} = 0.8$ $\rho_{1,3} = 0.8$
K	20	-50	104	-140	-30	35
T	1	1	1	1	1	1

Note: The notations are defined as follows: $S_i(0)$: the initial asset price; σ_i: volatility; α_i: units of the ith asset; $\rho_{i,j}$: correlation coefficient between S_i and S_j; K: strike price. The dividend yield rates of all assets are assumed to be zero, namely, $q_i = 0$ and the risk-free interest rate, r, is assumed to be 0.03.

Table 2 The Numerical Examples of *GB* Options Provided in Borovkova et al. (2007)

Method	*GB1*	*GB2*	*GB3*	*GB4*	*GB5*	*GB6*
USD	7.739	16.767	10.824	1.958	7.740	9.009
BPW	7.751	16.910	10.844	1.958	7.759	9.026
MC	7.744	16.757	10.821	1.966	7.730	9.012
se	0.014	0.023	0.018	0.005	0.010	0.015

Note: This table presents the pricing results of various *GB* options computed by three different approaches: USD represents the pricing model proposed in this article, BPW represents the pricing model presented in Borovkova et al. (2007), and MC denotes the Monte Carlo simulation method. The standard error of Monte Carlo simulation is denoted by se.

and Cathay Financial Holdings Co., Ltd. (2882). The market data of the representative companies include the stock price and dividend yield within the period from January 1, 2020, to August 31, 2022, and all data are from the Taiwan Economic Journal.

Assume that the valuation date is August 1, 2022; then, the initial stock of each company is $S_{2330}(0) = 505$, $S_{2603}(0) = 88.3$, and $S_{2882}(0) = 44.55$. The average yield of each company during this time period is $q_{2330} = 2.1\%$, $q_{2603} = 3.5\%$, and $q_{2882} = 4.8\%$. The strike price is assumed to be in-the-money. The historical volatility of each company is computed by the annualized standard deviation of stock return, which are $\sigma_{2330} = 27.4\%$, $\sigma_{2603} = 65.8\%$, and $\sigma_{2882} = 24.1\%$, respectively. The historical correlation coefficient between companies is calculated by the Pearson's correlation method, which are $\rho_{2330,2882} = 45.1\%$, $\rho_{2603,2882} = 34.0\%$, and $\rho_{2330,2603} = 17.1\%$, respectively. The risk-free interest rate, r,

Table 3 Values of Basket Calls (T = 1)

	ITM 10%, K = 135				*ATM*, K = 150				*OTM* 10%, K = 165			
ρ	MC	USD	SLN	SRG	MC	USD	SLN	SRG	MC	USD	SLN	SRG
Panel A: σ = 30%												
-0.7	23.01	22.94	23.46*	22.90	12.87	12.80	13.31*	12.85	6.49	6.46	6.56*	6.53
-0.5	23.98	23.91	24.24*	23.89	14.31	14.21	14.53*	14.26	7.81	7.75	7.85	7.82
-0.3	24.88	24.80	25.02	24.80	15.54	15.44	15.63	15.47	9.02	8.94	9.01	8.98
-0.1	25.72	25.64	25.77	25.64	16.63	16.54	16.65	16.55	10.12	10.04	10.09	10.05
0.1	26.51	26.42	26.50	26.43	17.63	17.54	17.60	17.54	11.15	11.06	11.09	11.05
0.3	27.25	27.17	27.21	27.18	18.56	18.47	18.50	18.46	12.11	12.02	12.04	12.00
0.5	27.97	27.88	27.90	27.89	19.43	19.34	19.36	19.32	13.02	12.93	12.94	12.90
0.7	28.65	28.57	28.57	28.57	20.26	20.17	20.18	20.15	13.88	13.80	13.80	13.76
Panel B: σ = 50%												
-0.7	27.44	27.20	28.97**	27.29	19.29	19.17	20.66**	19.31	13.43	13.42	14.31**	13.54
-0.5	29.46	29.18	30.36*	29.25	21.51	21.24*	22.30*	21.41	15.49	15.30*	16.04*	15.49
-0.3	31.19	30.91	31.69*	30.99	23.42	23.13*	23.86*	23.27	17.37	17.12*	17.67*	17.27
-0.1	32.75	32.49	32.98	32.57	25.15	24.87*	25.34	24.96	19.10	18.86*	19.22	18.93
0.1	34.18	33.95	34.24	34.01	26.73	26.49	26.76	26.53	20.72	20.50*	20.72	20.50*
0.3	35.52	35.31	35.46	35.37	28.22	28.00	28.14	28.00	22.26	22.05	22.17	22.00*
0.5	36.79	36.59	36.66	36.65	29.62	29.42	29.48	29.39	23.72	23.52	23.57	23.44*
0.7	38.00	37.81	37.83	37.88	30.96	30.76	30.78	30.74	25.12	24.93	24.94	24.82*
Panel C: σ = 70%												
-0.7	33.72	33.34*	36.91**	33.98	26.82	26.69	29.75***	27.16*	21.46	21.50	23.86***	21.78*
-0.5	36.35	35.76*	38.35**	36.21	29.47	28.97*	31.36**	29.43	23.93	23.54*	25.54**	23.94
-0.3	38.62	37.97*	39.82*	38.38	31.84	31.18*	32.97*	31.63	26.25	25.65*	27.24*	26.07
-0.1	40.67	40.06*	41.30*	40.44	34.01	33.36*	34.60*	33.74	28.43	27.80*	28.95*	28.15
0.1	42.58	42.05*	42.81	42.41	36.03	35.47*	36.25	35.78	30.49	29.94*	30.68	30.17*
0.3	44.37	43.92*	44.33	44.29	37.95	37.48*	37.90	37.73	32.47	32.01*	32.42	32.14*
0.5	46.07	45.69	45.87	46.11	39.78	39.39	39.58	39.63	34.38	33.99*	34.17	34.07
0.7	47.72	47.37	47.42	47.88	41.56	41.20	41.26	41.50	36.24	35.88	35.94	35.99

Note: The basket is composed of two assets, defined by $GB = S_1+S_2$. The relevant parameters are presented as follows: $S_1(0) = 100$, $S_2(0) = 50$, $q_1 = q_2 = 0$, $\sigma_1 = \sigma_2 = \sigma$, and $r = 0.05$. MC means the Monte Carlo methods, USD means the unbounded system distributions, SLN means the shifted lognormal distributions, and SRG means the shifted reciprocal gamma distributions. For ease of reading, the PPE of each case greater than 10% will be marked by ***; between 5% and 10% by **; and between 1% and 5% by *. No asterisk means that the PPE is lower than 1%. RMSE (USD, SLN, SRG) = (0.31, 0.82, 0.17) and MAE (USD, SLN, SRG) = (0.66, 3.18, 0.34).

Table 4 Values of Basket Calls (T = 1.5)

		ITM 10%, K = 135				*ATM*, K = 150				*OTM* 10%, K = 165			
		MC	USD	SLN	SRG	MC	USD	SLN	SRG	MC	USD	SLN	SRG
Panel A: σ = 30%													
ρ	-0.7	26.74	26.62	27.44*	26.57	16.91	16.81	17.66*	16.86	10.11	10.07	10.51*	10.17
	-0.5	27.97	27.85	28.38*	27.82	18.61	18.46	19.01*	18.53	11.79	11.68	12.01*	11.79
	-0.3	29.08	28.96	29.31	28.97	20.05	19.91	20.25	19.96	13.28	13.15	13.38	13.22
	-0.1	30.11	30.00	30.21	30.01	21.35	21.21	21.41	21.23	14.64	14.51	14.65	14.53
	0.1	31.08	30.96	31.09	30.98	22.53	22.40	22.52	22.40	15.88	15.76	15.84	15.75
	0.3	31.99	31.87	31.94	31.89	23.63	23.50	23.56	23.49	17.05	16.93	16.97	16.90
	0.5	32.86	32.74	32.77	32.76	24.66	24.54	24.57	24.52	18.16	18.04	18.05	17.99
	0.7	33.69	33.58	33.59	33.60	25.64	25.53	25.53	25.50	19.20	19.09	19.09	19.03
Panel B: σ = 50%													
ρ	-0.7	32.70	32.34*	35.09**	32.66	25.03	24.86	27.29**	25.14	19.16	19.15	20.98**	19.34
	-0.5	35.05	34.61*	36.51*	34.80	27.54	27.13*	28.93**	27.41	21.56	21.24*	22.72**	21.52
	-0.3	37.06	36.61*	37.91*	36.80	29.73	29.24*	30.53*	29.51	23.74	23.30*	24.42*	23.58
	-0.1	38.88	38.46*	39.31*	38.65	31.70	31.24*	32.10*	31.46	25.76	25.32*	26.09*	25.51
	0.1	40.55	40.18	40.69	40.37	33.53	33.13*	33.65	33.28	27.65	27.26*	27.74	27.36*
	0.3	42.12	41.80	42.07	42.00	35.25	34.91	35.18	35.02	29.44	29.11*	29.36	29.13*
	0.5	43.62	43.32	43.44	43.55	36.88	36.58	36.70	36.68	31.15	30.86	30.97	30.84*
	0.7	45.05	44.78	44.81	45.05	38.45	38.17	38.21	38.29	32.80	32.53	32.56	32.51
Panel C: σ = 70%													
ρ	-0.7	41.19	40.59*	45.97***	42.42*	34.90	34.61	39.48***	36.07*	29.78	29.71	33.91***	30.79*
	-0.5	43.99	43.11*	47.17**	44.45*	37.68	36.90*	40.79**	38.14*	32.40	31.76*	35.30**	32.81*
	-0.3	46.47	45.44*	48.47*	46.60	40.23	39.17*	42.20*	40.32	34.92	33.91*	36.78**	34.96
	-0.1	48.74	47.73*	49.85*	48.83	42.60	41.50*	43.71*	42.59	37.31	36.20*	38.37*	37.21
	0.1	50.86	49.99*	51.33	51.09	44.84	43.87*	45.31*	44.90	39.61	38.60*	40.06*	39.51
	0.3	52.87	52.17*	52.89	53.34	46.98	46.20*	47.01	47.21	41.82	41.01*	41.85	41.83
	0.5	54.81	54.23*	54.54	55.58*	49.05	48.43*	48.79	49.53	43.99	43.35*	43.72	44.18
	0.7	56.71	56.18	56.27	57.84*	51.08	50.54*	50.65	51.87*	46.12	45.57*	45.68	46.57

Note: The basket is composed of two assets, defined by $GB = S_1+S_2$. The relevant parameters are presented as follows: $S_1(0)$ = 100, $S_2(0)$ = 50, $q_1 = q_2 = 0$, $\sigma_1 = \sigma_2 = \sigma$, and r = 0.05. MC means the Monte Carlo methods, USD means the unbounded system distributions, SLN means the shifted lognormal distributions, and SRG means the shifted reciprocal gamma distributions. For ease of reading, the PPE of each case greater than 10% will be marked by ***; between 5% and 10% by **; and between 1% and 5% by *. No asterisk means that the PPE is lower than 1%. RMSE (USD, SLN, SRG) = (0.51, 1.34, 0.36) and MAE (USD, SLN, SRG) = (1.11, 4.78, 1.23).

Table 5 Values of Basket Calls (T = 2)

		ITM 10%, K = 135				*ATM*, K = 150				*OTM* 10%, K = 165			
		MC	USD	SLN	SRG	MC	USD	SLN	SRG	MC	USD	SLN	SRG
Panel A: σ = 30%													
ρ	-0.7	30.24	30.07	31.20*	30.02	20.64	20.50	21.70**	20.57	13.61	13.55	14.37**	13.66
	-0.5	31.65	31.49	32.23*	31.46	22.53	22.33	23.12*	22.41	15.55	15.38	15.98*	15.52
	-0.3	32.92	32.76	33.24	32.76	24.13	23.93	24.44*	24.00	17.25	17.06	17.46*	17.16
	-0.1	34.09	33.93	34.24	33.96	25.57	25.38	25.69	25.43	18.78	18.60	18.85	18.64
	0.1	35.19	35.03	35.21	35.07	26.88	26.71	26.89	26.73	20.19	20.02	20.17	20.02
	0.3	36.22	36.07	36.16	36.11	28.11	27.95	28.04	27.95	21.51	21.35	21.43	21.32
	0.5	37.21	37.06	37.10	37.10	29.26	29.11	29.15	29.11	22.76	22.61	22.64	22.55
	0.7	38.15	38.00	38.02	38.06	30.36	30.22	30.23	30.21	23.94	23.80	23.81	23.73
Panel B: σ = 50%													
ρ	-0.7	37.54	37.03*	40.78**	37.76	30.25	29.99	33.41***	30.59*	24.46	24.42	27.24***	24.82*
	-0.5	40.08	39.48*	42.12**	39.91	32.93	32.37	34.94**	32.85	27.06	26.60*	28.88**	27.05
	-0.3	42.27	41.64*	43.50*	42.02	35.28	34.60	36.49*	35.06	29.43	28.78*	30.54*	29.24
	-0.1	44.26	43.67*	44.90*	44.05	37.42	36.76	38.06*	37.18	31.62	30.96*	32.21*	31.35
	0.1	46.09	45.58*	46.33	45.98	39.41	38.84	39.64	39.20	33.69	33.11*	33.90	33.40
	0.3	47.82	47.38	47.78	47.82	41.29	40.82	41.25	41.15	35.65	35.18*	35.61	35.39
	0.5	49.47	49.08	49.26	49.60	43.08	42.68	42.87	43.03	37.55	37.14*	37.33	37.33
	0.7	51.06	50.70	50.75	51.34	44.81	44.45	44.51	44.88	39.38	39.01*	39.07	39.24
Panel C: σ = 70%													
ρ	-0.7	47.98	47.14*	54.15***	50.86**	42.17	41.65*	48.21***	44.87**	37.32	37.03	42.99***	39.71**
	-0.5	50.79	49.60*	55.07**	52.59*	44.94	43.86*	49.20**	46.62*	39.94	39.00*	44.05***	41.45*
	-0.3	53.34	51.93*	56.12**	54.56*	47.54	46.07*	50.35**	48.63*	42.51	41.07*	45.26**	43.44*
	-0.1	55.71	54.30*	57.32*	56.77*	50.00	48.43*	51.64*	50.87*	45.00	43.36*	46.63*	45.67*
	0.1	57.94	56.72*	58.67*	59.17*	52.35	50.93*	53.10*	53.31*	47.41	45.89*	48.16*	48.11*
	0.3	60.08	59.13*	60.17	61.72*	54.61	53.50*	54.72	55.91*	49.75	48.54*	49.87	50.72*
	0.5	62.15	61.40*	61.82	64.37*	56.82	55.98*	56.50	58.63*	52.06	51.16*	51.75	53.47*
	0.7	64.20	63.51*	63.62	67.14*	59.01	58.29*	58.43	61.48*	54.37	53.62*	53.79*	56.37*

Note: The basket is composed of two assets, defined by $GB = S_1+S_2$. The relevant parameters are presented as follows: $S_1(0) = 100$, $S_2(0) = 50$, $q_1 = q_2 = 0$, $\sigma_1 = \sigma_2 = \sigma$, and $r = 0.05$. MC means the Monte Carlo methods, USD means the unbounded system distributions, SLN means the shifted lognormal distributions, and SRG means the shifted reciprocal gamma distributions. For ease of reading, the PPE of each case greater than 10% will be marked by ***; between 5% and 10% by **; and between 1% and 5% by *. No asterisk means that the PPE is lower than 1%. RMSE (USD, SLN, SRG) = (0.72, 1.84, 1.01) and MAE (USD, SLN, SRG) = (1.63, 6.17, 2.94).

Table 6 Values of Spread Calls (T = 1)

		ITM 10%, K = 45				*ATM*, K = 50				*OTM* 10%, K = 55			
		MC	USD	SLN	SRG	MC	USD	SLN	SRG	MC	USD	SLN	SRG
Panel A: σ = 30%													
ρ	-0.7	21.40	21.38	21.60	21.59	18.93	18.91	19.15*	19.14*	16.67	16.65	16.90*	16.89*
	-0.5	20.55	20.54	20.75	20.74	18.08	18.06	18.30*	18.29*	15.82	15.81	16.06*	16.04*
	-0.3	19.65	19.64	19.84	19.83	17.17	17.16	17.39*	17.37*	14.93	14.91	15.15*	15.14*
	-0.1	18.68	18.68	18.86	18.85	16.20	16.19	16.40*	16.38*	13.96	13.95	14.17*	14.15*
	0.1	17.64	17.64	17.79	17.78	15.14	15.14	15.31*	15.30*	12.92	12.91	13.10*	13.08*
	0.3	16.50	16.51	16.61	16.60	13.99	13.98	14.12	14.10	11.77	11.76	11.91*	11.89*
	0.5	15.23	15.24	15.30	15.29	12.69	12.69	12.77	12.76	10.48	10.47	10.57	10.55
	0.7	13.80	13.79	13.81	13.81	11.20	11.19	11.23	11.21	9.00	8.98	9.03	9.01
Panel B: σ = 50%													
ρ	-0.7	31.62	31.63	32.61*	32.57*	29.35	29.35	30.38*	30.34*	27.21	27.20	28.27*	28.23*
	-0.5	30.28	30.35	31.28*	31.23*	28.02	28.07	29.06*	29.01*	25.90	25.93	26.97*	26.92*
	-0.3	28.84	28.95	29.81*	29.76*	26.59	26.68	27.61*	27.55*	24.50	24.56	25.54*	25.48*
	-0.1	27.29	27.42	28.17*	28.13*	25.05	25.16	25.99*	25.94*	22.98	23.06	23.94*	23.88*
	0.1	25.58	25.74	26.35*	26.31*	23.36	23.49	24.18*	24.13*	21.32	21.41	22.16*	22.09*
	0.3	23.69	23.85	24.29*	24.26*	21.49	21.61	22.13*	22.08*	19.47	19.56	20.13*	20.07*
	0.5	21.55	21.68	21.93*	21.93*	19.36	19.46	19.78*	19.75*	17.38	17.45	17.81*	17.76*
	0.7	19.06	19.13	19.20	19.24	16.87	16.92	17.04	17.04	14.92	14.95	15.10*	15.07
Panel C: σ = 70%													
ρ	-0.7	41.22	41.38	43.75**	43.66**	39.08	39.21	41.71**	41.61**	37.05	37.15	39.75**	39.63**
	-0.5	39.53	39.87	42.06**	41.99**	37.42	37.72	40.04**	39.95**	35.42	35.67	38.11**	38.00**
	-0.3	37.70	38.17*	40.14**	40.09**	35.61	36.03*	38.15**	38.07**	33.65	34.01*	36.24**	36.14**
	-0.1	35.68	36.25*	37.94**	37.92**	33.63	34.14*	35.97**	35.92**	31.70	32.14*	34.09**	34.02**
	0.1	33.44	34.07*	35.40**	35.44**	31.42	31.98*	33.46**	33.45**	29.52	30.01*	31.62**	31.58**
	0.3	30.91	31.54*	32.46**	32.57**	28.92	29.47*	30.54**	30.61**	27.07	27.55*	28.74**	28.76**
	0.5	27.99	28.54*	29.02*	29.24*	26.04	26.52*	27.13*	27.28*	24.24	24.64*	25.37*	25.46**
	0.7	24.51	24.85*	24.95*	25.29*	22.59	22.88*	23.07*	23.33*	20.85	21.09*	21.35*	21.53*

Note: The basket is composed of two assets, defined by $GB = S_1 - S_2$. The relevant parameters are presented as follows: $S_1(0) = 100$, $S_2(0) = 50$, $q_1 = q_2 = 0$, $\sigma_1 = \sigma_2 = \sigma$, and $r = 0.05$. MC means the Monte Carlo methods, USD means the unbounded system distributions, SLN means the shifted lognormal distributions, and SRG means the shifted reciprocal gamma distributions. For ease of reading, the PPE of each case greater than 10% will be marked by ***; between 5% and 10% by **; and between 1% and 5% by *. No asterisk means that the PPE is lower than 1%. RMSE (USD, SLN, SRG) = (0.25, 1.28, 1.26) and MAE (USD, SLN, SRG) = (0.63, 2.70, 2.58).

Table 7 Values of Spread Calls (T = 1.5)

		ITM 10%, K = 45				*ATM*, K = 50				*OTM* 10%, K = 55			
		MC	USD	SLN	SRG	MC	USD	SLN	SRG	MC	USD	SLN	SRG
		Panel A: σ = 30%											
ρ	-0.7	26.00	25.98	26.36*	26.35*	23.65	23.63	24.05*	24.03*	21.45	21.43	21.87*	21.86*
	-0.5	24.97	24.97	25.34*	25.32*	22.62	22.61	23.02*	23.00*	20.43	20.42	20.86*	20.84*
	-0.3	23.87	23.89	24.23*	24.22*	21.52	21.53	21.91*	21.89*	19.34	19.34	19.75*	19.73*
	-0.1	22.70	22.73	23.03*	23.01*	20.34	20.36	20.70*	20.68*	18.17	18.18	18.55*	18.53*
	0.1	21.43	21.47	21.70*	21.69*	19.06	19.08	19.37*	19.35*	16.90	16.91	17.23*	17.20*
	0.3	20.03	20.07	20.24*	20.23	17.65	17.68	17.89*	17.87*	15.49	15.50	15.75*	15.72*
	0.5	18.48	18.51	18.60	18.59	16.07	16.09	16.22	16.20	13.91	13.92	14.08*	14.05*
	0.7	16.71	16.72	16.74	16.75	14.24	14.25	14.30	14.29	12.08	12.08	12.15	12.12
		Panel B: σ = 50%											
ρ	-0.7	38.11	38.19	39.84*	39.78*	35.97	36.04	37.78**	37.71*	33.94	33.99	35.81**	35.72**
	-0.5	36.55	36.75	38.29*	38.22*	34.42	34.60	36.24**	36.17**	32.41	32.56	34.29**	34.20**
	-0.3	34.86	35.15	36.54*	36.48*	32.75	33.01	34.51**	34.44**	30.77	30.99	32.58**	32.49**
	-0.1	33.02	33.38*	34.57*	34.53*	30.93	31.25*	32.55**	32.49**	28.97	29.24	30.64**	30.57**
	0.1	30.99	31.39*	32.32*	32.31*	28.92	29.27*	30.33*	30.29*	26.99	27.28*	28.44**	28.38**
	0.3	28.72	29.11*	29.76*	29.79*	26.67	27.02*	27.78*	27.78*	24.77	25.06*	25.92*	25.89*
	0.5	26.12	26.47*	26.80*	26.89*	24.09	24.39*	24.82*	24.87*	22.22	22.47*	22.99*	23.00*
	0.7	23.07	23.26	23.33*	23.51*	21.04	21.21	21.35*	21.47*	19.21	19.34	19.54*	19.60*
		Panel C: σ = 70%											
ρ	-0.7	49.11	49.47	53.03**	53.08**	47.13	47.45	51.21**	51.23**	45.23	45.51	49.46**	49.43**
	-0.5	47.25	47.96*	51.19**	51.31**	45.29	45.96*	49.40**	49.47**	43.43	44.04*	47.67**	47.70**
	-0.3	45.20	46.20*	49.04**	49.25**	43.28	44.21*	47.26**	47.42**	41.45	42.31*	45.56**	45.66***
	-0.1	42.91	44.13*	46.50**	46.82**	41.02	42.16*	44.74**	45.01**	39.24	40.28*	43.06**	43.27***
	0.1	40.33	41.68*	43.49**	43.97**	38.48	39.73*	41.76**	42.17**	36.74	37.88*	40.11**	40.45***
	0.3	37.36	38.73*	39.91**	40.60**	35.55	36.81*	38.21**	38.81**	33.86	35.00*	36.59**	37.10**
	0.5	33.89	35.11*	35.64**	36.57**	32.12	33.24*	33.96**	34.78**	30.48	31.49*	32.37**	33.09**
	0.7	29.66	30.49*	30.47*	31.66**	27.94	28.69*	28.80*	29.86**	26.35	27.02*	27.25*	28.18**

Note: The basket is composed of two assets, defined by $GB = S_1 - S_2$. The relevant parameters are presented as follows: $S_1(0) = 100$, $S_2(0) = 50$, $q_1 = q_2 = 0$, $\sigma_1 = \sigma_2 = \sigma$, and $r = 0.05$. MC means the Monte Carlo methods, USD means the unbounded system distributions, SLN means the shifted lognormal distributions, and SRG means the shifted reciprocal gamma distributions. For ease of reading, the PPE of each case greater than 10% will be marked by ***; between 5% and 10% by **; and between 1% and 5% by *. No asterisk means that the PPE is lower than 1%. RMSE (USD, SLN, SRG) = (0.59, 2.06, 2.21) and MAE (USD, SLN, SRG) = (1.37, 4.23, 4.26).

Table 8 Values of Spread Calls (T = 2)

		ITM 10%, K = 45				*ATM*, K = 50				*OTM* 10%, K = 55			
		MC	USD	SLN	SRG	MC	USD	SLN	SRG	MC	USD	SLN	SRG
Panel A: σ = 30%													
	-0.7	29.93	29.92	30.49*	30.47*	27.69	27.68	28.29*	28.27*	25.58	25.56	26.21*	26.18*
	-0.5	28.76	28.78	29.32*	29.30*	26.52	26.54	27.13*	27.10*	24.41	24.42	25.05*	25.02*
	-0.3	27.51	27.56	28.05*	28.03*	25.27	25.31	25.86*	25.83*	23.17	23.20	23.79*	23.76*
ρ	-0.1	26.17	26.24	26.66*	26.64*	23.92	23.98	24.46*	24.44*	21.83	21.87	22.40*	22.37*
	0.1	24.71	24.79	25.13*	25.11*	22.46	22.52	22.92*	22.89*	20.37	20.42	20.87*	20.83*
	0.3	23.10	23.19	23.42*	23.41*	20.84	20.91	21.19*	21.17*	18.75	18.80	19.14*	19.11*
	0.5	21.31	21.38	21.49	21.49	19.02	19.07	19.24*	19.23*	16.93	16.97	17.18*	17.16*
	0.7	19.25	19.29	19.30	19.33	16.91	16.94	16.99	17.00	14.81	14.83	14.91	14.89
Panel B: σ = 50%													
	-0.7	43.46	43.64	45.94**	45.87**	41.44	41.60	44.04**	43.95**	39.51	39.65	42.20**	42.10**
	-0.5	41.75	42.12	44.24**	44.18**	39.74	40.09	42.34**	42.27**	37.84	38.15	40.53**	40.43**
	-0.3	39.88	40.41*	42.28**	42.26**	37.90	38.39*	40.41**	40.36**	36.02	36.46*	38.61**	38.54**
ρ	-0.1	37.83	38.48*	40.05**	40.07**	35.87	36.46*	38.19**	38.18**	34.02	34.55*	36.42**	36.37**
	0.1	35.55	36.26*	37.47**	37.55**	33.61	34.27*	35.63**	35.68**	31.79	32.37*	33.88**	33.89**
	0.3	32.98	33.70*	34.49*	34.66**	31.06	31.72*	32.66**	32.79**	29.27	29.85*	30.94**	31.01**
	0.5	30.02	30.65*	31.00*	31.30*	28.12	28.69*	29.19*	29.42*	26.36	26.87*	27.48*	27.65*
	0.7	26.49	26.88*	26.89*	27.33*	24.61	24.96*	25.07*	25.43*	22.88	23.19*	23.37*	23.66*
Panel C: σ = 70%													
	-0.7	55.35	55.89	60.17**	60.89***	53.50	54.00	58.56**	59.21***	51.73	52.18	56.99***	57.57***
	-0.5	53.41	54.51*	58.36**	59.23***	51.60	52.63*	56.77***	57.55***	49.87	50.83*	55.22***	55.93***
	-0.3	51.25	52.82*	56.16**	57.22***	49.48	50.96*	54.59***	55.55***	47.79	49.17*	53.06***	53.93***
ρ	-0.1	48.81	50.74*	53.49**	54.77***	47.07	48.89*	51.93***	53.11***	45.42	47.13*	50.42***	51.51***
	0.1	46.00	48.17*	50.22**	51.79***	44.30	46.35*	48.68**	50.14***	42.69	44.61*	47.20***	48.55***
	0.3	42.73	44.97**	46.23**	48.15***	41.07	43.18**	44.72**	46.51***	39.51	41.47*	43.26**	44.93***
	0.5	38.83	40.89**	41.33**	43.65***	37.23	39.15**	39.83**	42.00***	35.72	37.50*	38.40**	40.43***
	0.7	34.00	35.48*	35.25*	37.94***	32.44	33.81*	33.76*	36.27***	30.99	32.25*	32.36*	34.70***

Note: The basket is composed of two assets, defined by $GB = S_1 - S_2$. The relevant parameters are presented as follows: $S_1(0) = 100$, $S_2(0) = 50$, $q_1 = q_2 = 0$, $\sigma_1 = \sigma_2 = \sigma$, and $r = 0.05$. MC means the Monte Carlo methods, USD means the unbounded system distributions, SLN means the shifted lognormal distributions, and SRG means the shifted reciprocal gamma distributions. For ease of reading, the PPE of each case greater than 10% will be marked by ***; between 5% and 10% by **; and between 1% and 5% by *. No asterisk means that the PPE is lower than 1%. RMSE (USD, SLN, SRG) = (0.98, 2.72, 3.40) and MAE (USD, SLN, SRG) = (2.24, 5.35, 6.15).

is assumed to be 1.5%. The weight of each company makes the underlying general basket belong to the basket options if all weights are positive, including *GB1*, *GB2*, and *GB4*, and belong to the spread options if some weights are negative, including *GB3*, *GB5*, and *GB6*. All parameters are summarized in Table 9.

Table 10 presents the pricing results of various *GB* computed by three different pricing methods: USD represents the pricing model based on the unbounded system of the Johnson distribution family, SLN represents the pricing model developed by Borovkova et al. (2007), and MC denotes the Monte Carlo simulation method based on 100,000 simulation paths with the variance reduction technique named the antithetic variates method. The standard error of the Monte Carlo simulation method is denoted by "se". All pricing results show that our pricing model can accurately price both the basket and spread options based on the market data.

4.3. Sensitivity Analysis

Since basket and spread options do not have close-form pricing models, their pricing models in the early literature are developed independently under various model assumptions. This may lead to inconsistency, and cause pricing and hedging errors between basket and spread options. However, our pricing model can price both basket and spread options, and thus, it can eliminate the pricing errors. In addition, the Greeks of basket and spread options are derived from the same pricing formulas; in consequence their Greek risks can be integrated to help traders manage and hedge their option portfolios.

As indicated by Figures 4, 5, and 6, the correlation coefficient ρ substantially affects the Greeks of both basket and spread options. The humped-shape figure of the *correlation vega* shows that ρ positively affects the basket-option value, which decreases with increasing ρ. On the contrary, ρ negatively affects the spread-option value, which increases with decreasing ρ. The behavior of *vega* (υ) and *delta* (Δ) of an asset is affected by ρ, moneyness, and the (long or short) position of the asset. These Greeks can help financial institutions construct hedging strategies to manage the risks of issuing basket/spread options.

Next, we present some numerical examples to demonstrate the sensitivity analysis of basket and spread options based on the Greek formulas provided in Definition 2. To save space, we only show the *delta* (Δ), *vega* (υ), and *correlation vega* of both options in Figures 4, 5, and 6. Other Greeks can also be easily examined by using Definition 2. For

Table 9 The *GB* Parameters of Each Numerical Example with Market Data

Terms	*GB1*	*GB2*	*GB3*	*GB4*	*GB5*	*GB6*
Stock ID	[2330, 2882]	[2603, 2882]	[2330, 2603]	[2330, 2603, 2882]		
$S_i(0)$	[505, 44.55]	[88.3, 44.55]	[505, 88.3]	[505, 88.3, 44.55]		
σ_i	[0.274, 0.241]	[0.658, 0.241]	[0.274, 0.658]	[0.274, 0.658, 0.241]		
α_i	[1, 1]	[1, 1]	[1, -1]	[1, 1, 1]	[1, -1, 1]	[1, -1, -1]
q_i	[0.021, 0.048]	[0.035, 0.048]	[0.021, 0.035]	[0.021, 0.035, 0.048]		
$\rho_{i,j}$	$\rho_{1,2}$ = 0.451	$\rho_{1,2}$ = 0.340	$\rho_{1,2}$ = 0.171	$\rho_{1,2}$ = 0.171, $\rho_{1,3}$ = 0.451, $\rho_{2,3}$ = 0.340,		
K	549.55	132.85	593.3	637.85	461.25	372.15

Note: The notations are defined as follows: $S_i(0)$: the initial asset price; σ_i: volatility; α_i: units of the ith asset; q_i: dividend yield rate; $\rho_{i,j}$: correlation coefficient between S_i and S_j; K: strike price. The time to maturity, T, is assumed to be 1. The risk-free interest rate, r, is assumed to be 0.015.

Table 10 The *GB* Parameters of Each Numerical Example with Market Data

Methods	*GB1*	*GB2*	*GB3*	*GB4*	*GB5*	*GB6*
USD	54.094	22.445	54.584	61.910	55.368	54.086
SLN	54.155	23.540	55.977	62.246	56.615	55.631
MC	54.077	22.487	54.543	62.122	55.327	54.079
se	0.213	0.114	0.207	0.247	0.212	0.204

Note: This table presents the pricing results of various *GB* options computed by three different approaches: USD represents the pricing model proposed in this article, SLN represents the pricing model presented in Borovkova et al. (2007), and MC denotes the Monte Carlo simulation method. The standard error of Monte Carlo simulation is denoted by se.

simplicity, we assume that both basket and spread options are composed of two assets, and their parameters are given in the footnotes of Figures 4, 5, and 6.

Figure 6 provides numerical examples, which show the Greeks of an option portfolio composed of a long position in a basket option on *GB7* and a short position in a spread option on *GB8* with the same parameters defined in the footnotes of Figures 4 and 5. Notably, the patterns of the Greeks of the option portfolio are totally different from those of a single basket or spread option, and are not easily understood simply via economic intuitions. This fact reveals the importance of our pricing model for integrating the Greek risks of both options, which enhances hedging efficiency and reduces the cost for hedging option portfolios.

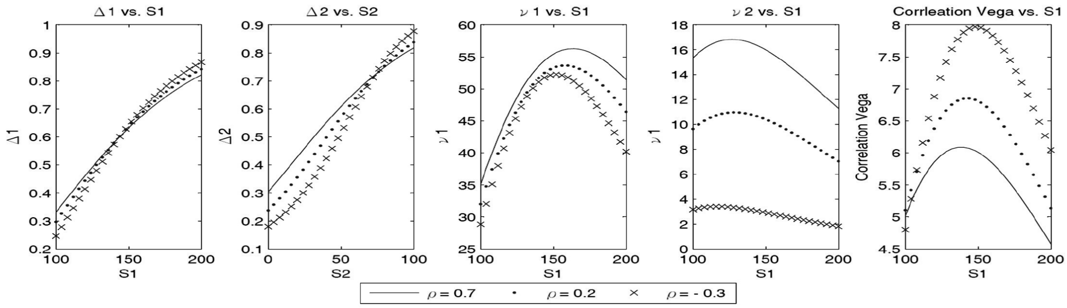

Figure 4 Greeks of Basket Options

Note: The basket is defined by $GB7 = S_1+S_2$, and the other parameters are defined as follows: $S_1(0) = 150$, $S_2(0) = 50$, $\sigma_1 = \sigma_2 = 0.4$, $K = 200$, $r = 0.05$, and $T = 1$.

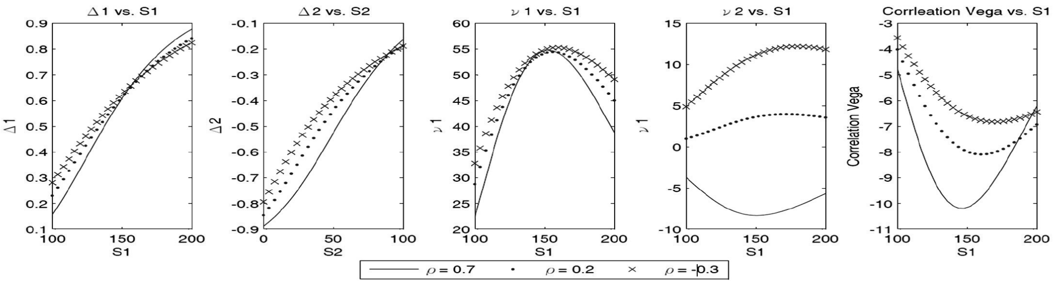

Figure 5 Greeks of Spread Options

Note: The spread is defined by $GB8 = S_1–S_2$, and the other parameters are defined as follows: $S_1(0) = 150$, $S_2(0) = 50$, $\sigma_1 = \sigma_2 = 0.4$, $K = 200$, $r = 0.05$, and $T = 1$.

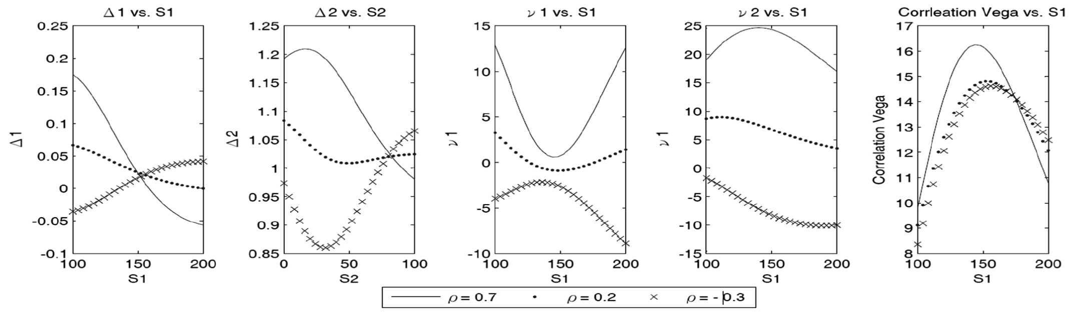

Figure 6 Greeks of an Option Portfolio

Note: The option portfolio is composed of a long position in a basket option by *GB7* and a short position in a spread option on *GB8* with parameters defined in Figures 4 and 5.

5. Conclusion

This study adopts the unbounded-system distribution of the Johnson (1949) distribution family to approximate the basket/spread distribution, and derive a united pricing model for both basket and spread options. Our proposed pricing model can accurately and instantly price both basket and spread options even in difficult situations, where option maturity is longer and underlying assets exhibit high volatilities and low correlation. Besides the pricing advantage, the Greeks derived from our pricing formulas help financial institutions efficiently integrate and manage the risks of issuing both basket and spread options. Therefore, our pricing model can reduce pricing errors, enhance hedging efficiency; thus lower the hedging cost of both basket and spread options. Based on the aforementioned merits, the resulting pricing formulas provide market practitioners with an accurate, efficient and time-saving approach for offering almost instantly-quoted prices to clients and the daily marking-to-market trading books, and facilitating efficient risk management of trading positions. Thus, the presented formulas are worth recommending to market practitioners.[11]

11 This study adopts a geometric Brownian motion to specify the dynamics of the prices of the underlying assets. Future research can employ the stochastic volatility model or a jump diffusion process to specify these dynamics.

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Appendix A. The Pairs of (SK^2, K) of the GB Distribution

Based on the parameters we used from Tables 3 to 8, the pair of (SK^2, K) of the *GB* distribution can be computed by Proposition 1 and equations (1) to (4). All numerical results are shown in Figure 7.

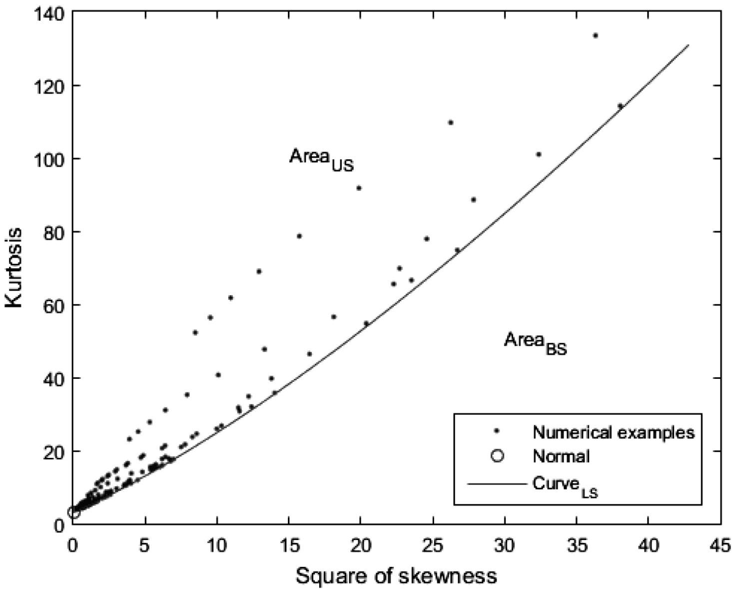

Figure 7 The pair of (SK^2, K) of the GB Distribution

Appendix B. Proof of Proposition 2

Based on the definition in equation (8), $X = a + bY$, and $Y = sinh\left(\frac{z - c}{d}\right)$, where Z is a standard normal random variable, $a \in R$, $b > 0$, $c \in R$, and $d > 0$. The four characteristics of the US distribution given in Proposition 2 can be obtained by the following derivations.

- The mean of X ($\mathcal{M}_{\mathrm{US}}(a,b,c,d)$)

$$\mathcal{M}_{\mathrm{US}}(a,b,c,d) = \mathrm{E}(X) = a + b\mathrm{E}(Y) = a - b\omega^{\frac{1}{2}}\sinh(\Omega),$$

where

$$\begin{aligned}
\mathrm{E}(Y) &= \mathrm{E}\left[\frac{1}{2}\exp\left(\frac{Z-c}{d}\right) - \frac{1}{2}\exp\left(-\frac{Z-c}{d}\right)\right] \\
&= \frac{1}{2}\exp\left(\frac{-c}{d}\right)\mathrm{E}\left[\exp\left(\frac{Z}{d}\right)\right] - \frac{1}{2}\exp\left(\frac{c}{d}\right)\mathrm{E}\left[\exp\left(\frac{-Z}{d}\right)\right] \\
&= \frac{1}{2}e^{-\Omega}\omega^{\frac{1}{2}} - \frac{1}{2}e^{\Omega}\omega^{\frac{1}{2}} \\
&= \frac{1}{2}\omega^{\frac{1}{2}}(e^{-\Omega} - e^{\Omega}) \\
&= -\omega^{\frac{1}{2}}\sinh(\Omega),
\end{aligned}$$

$\Omega = c/d$, and $\omega = \exp(1/d^2)$.

- The variance of X ($\mathcal{V}_{\mathrm{US}}(a,b,c,d)$)

$$\begin{aligned}
\mathcal{V}_{\mathrm{US}}(a,b,c,d) &= \mathrm{V}(X) \\
&= b^2\mathrm{V}(Y) \\
&= b^2[E(Y^2) - E(Y)^2] \\
&= b^2\left[\frac{1}{2}(\omega^2\cosh(2\Omega) - 1) - \omega\sinh^2(\Omega)\right] \\
&= b^2\left[\frac{1}{2}(\omega^2\cosh(2\Omega) - 1) - \frac{\omega}{2}[\cosh(2\Omega) - 1]\right] \\
&= \frac{b^2}{2}(\omega - 1)(\omega\cosh(2\Omega) + 1),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{E}(Y^2) &= \mathrm{E}\left\{\left[\frac{1}{2}\exp\left(\frac{Z-c}{d}\right) - \frac{1}{2}\exp\left(-\frac{Z-c}{d}\right)\right]^2\right\} \\
&= \frac{1}{4}\exp\left(\frac{-2c}{d}\right)\mathrm{E}\left[\exp\left(\frac{2Z}{d}\right)\right] - \frac{1}{4}\exp\left(\frac{2c}{d}\right)\mathrm{E}\left[\exp\left(\frac{-2Z}{d}\right)\right] - \frac{1}{2} \\
&= \frac{1}{4}e^{-2\Omega}\omega^2 + \frac{1}{4}e^{2\Omega}\omega^2 - \frac{1}{2} \\
&= \frac{1}{2}(\omega^2\cosh(2\Omega) - 1).
\end{aligned}$$

- The skewness of X ($\mathcal{SK}_{\mathrm{US}}(a,b,c,d)$)

$$\mathcal{SK}_{\mathrm{US}}(a,b,c,d) = \frac{E\big(X - E(X)\big)^3}{[\mathrm{V}(X)]^{\frac{3}{2}}}$$

$$= \frac{E(Y^3) - 3E(Y^2)E(Y) + 2E^3(Y)}{[\mathrm{V}(Y)]^{\frac{3}{2}}}$$

$$= \frac{-\sqrt{\omega(\omega-1)}[\omega(\omega+2)\sinh(3\Omega) + 3\sinh(\Omega)]}{\sqrt{2[\omega\cosh(2\Omega)+1]^3}},$$

where the numerator part of the skewness is

$$\mathrm{E}(Y^3) - 3\mathrm{E}(Y^2)\mathrm{E}(Y) + 2E^3(Y)$$

$$= \frac{-1}{4}\omega^{\frac{9}{2}}\sinh(3\Omega) + \frac{3}{4}\omega^{\frac{1}{2}}\sinh(\Omega) + \frac{3}{2}(\omega^2\cosh(2\Omega) - 1)\left(\omega^{\frac{1}{2}}\sinh(\Omega)\right)$$

$$- 2\omega^{\frac{3}{2}}\sinh^3(\Omega)$$

$$= \frac{3}{4}\omega^{\frac{1}{2}}\sinh(\Omega) - \frac{1}{4}\omega^{\frac{9}{2}}\sinh(3\Omega) + \frac{3}{4}\omega^{\frac{5}{2}}\sinh(3\Omega) - \frac{3}{4}\omega^{\frac{5}{2}}\sinh(\Omega) - \frac{3}{2}\omega^{\frac{1}{2}}\sinh(\Omega)$$

$$- \frac{1}{2}\omega^{\frac{3}{2}}\sinh(3\Omega) + \frac{3}{2}\omega^{\frac{3}{2}}\sinh(\Omega)$$

$$= -\frac{1}{4}\sinh(3\Omega)\omega^{\frac{3}{2}}(\omega^3 - 3\omega + 2) - \frac{3}{4}\sinh(\Omega)\omega^{\frac{1}{2}}(\omega^2 - 2\omega - 1)$$

$$= -\frac{1}{4}\omega^{\frac{1}{2}}(\omega-1)^2[\omega(\omega+2)\sinh(3\Omega) + 3\sinh(\Omega)],$$

where

$$\sinh^3(\Omega) = \frac{1}{4}[\sinh(3\Omega) - 3\sinh(\Omega)],$$

$$\sinh(\Omega)\cosh(2\Omega) = \frac{1}{2}[\sinh(3\Omega) - \sinh(\Omega)],$$

and

$$\mathrm{E}(Y^3) = \mathrm{E}\left\{\left[\frac{1}{2}\exp\left(\frac{Z-c}{d}\right) - \frac{1}{2}\exp\left(-\frac{Z-c}{d}\right)\right]^3\right\}$$

$$= \frac{1}{8}\exp\left(\frac{-3c}{d}\right)\mathrm{E}\left[\exp\left(\frac{3Z}{d}\right)\right] - \frac{3}{8}\exp\left(\frac{-c}{d}\right)\mathrm{E}\left[\exp\left(\frac{Z}{d}\right)\right] + \frac{3}{8}\exp\left(\frac{c}{d}\right)\mathrm{E}\left[\exp\left(\frac{-Z}{d}\right)\right]$$

$$- \frac{1}{8}\exp\left(\frac{3c}{d}\right)\mathrm{E}\left[\exp\left(\frac{-3Z}{d}\right)\right]$$

$$= \frac{1}{8}e^{-3\Omega}\omega^{\frac{9}{2}} - \frac{3}{8}e^{-\Omega}\omega^{\frac{1}{2}} + \frac{3}{8}e^{\Omega}\omega^{\frac{9}{2}} - \frac{1}{8}e^{3\Omega}\omega^{\frac{9}{2}}$$

$$= \frac{-1}{4}\omega^{\frac{9}{2}}\sinh(3\Omega) + \frac{3}{4}\omega^{\frac{1}{2}}\sinh(\Omega),$$

$\Omega = c/d$, and $\omega = \exp(1/d^2)$.

- The kurtosis of X ($\mathcal{K}_{\mathrm{US}}(a,b,c,d)$)

$$\mathcal{K}_{\mathrm{US}}(a,b,c,d) = \frac{E\big(X - E(X)\big)^4}{[\mathrm{V}(X)]^2}$$

$$= \frac{E(Y^4) - 4E(Y^3)E(Y) + 6E(Y^2)E^2(Y) - 3E^4(Y)}{[\mathrm{V}(Y)]^{\frac{3}{2}}}$$

$$= \frac{\omega^2(\omega^4 + 2\omega^3 + 3\omega^2 - 3)\cosh(4\Omega) + 4\omega^2(\omega + 2)\cosh(2\Omega) + 3(2\omega + 1)}{2[\omega\cosh(2\Omega) + 1]^2},$$

where the numerator part of the skewness is

$$E(Y^4) - 4E(Y^3)E(Y) + 6E(Y^2)E^2(Y) - 3E^4(Y)$$

$$= \frac{1}{8}\omega^8\cosh(4\Omega) - \frac{1}{2}\omega^2\cosh(2\Omega) + \frac{3}{8} - \omega^5\sinh(3\Omega)\sinh(\Omega) + 3\omega\sinh^2(\Omega)$$

$$+ 3\omega^3\sinh^2(\Omega)\cosh(2\Omega) - 3\omega\sinh^2(\Omega) - 3\omega^2\sinh^4(\Omega)$$

$$= \frac{1}{8}\omega^8\cosh(4\Omega) - \frac{1}{2}\omega^2\cosh(2\Omega) + \frac{3}{8} - \frac{1}{2}\omega^5\cosh(4\Omega) + \frac{1}{2}\omega^5\cosh(2\Omega)$$

$$+ \frac{3}{2}\omega\cosh(2\Omega) - \frac{3}{2}\omega - \frac{3}{4}\omega^3\cosh(4\Omega) - \frac{3}{2}\omega^3\cosh(2\Omega) + \frac{3}{4}\omega^3$$

$$- \frac{3}{2}\omega\cosh(2\Omega) + \frac{3}{2}\omega - \frac{3}{8}\omega^2\cosh(4\Omega) + \frac{3}{2}\omega^2\cosh(2\Omega) - \frac{9}{8}\omega^2$$

$$= \frac{1}{8}\omega^2\cosh(4\Omega)(\omega^6 - 4\omega^3 + 6\omega - 3) + \frac{1}{2}\omega^2\cosh(2\Omega)(\omega^3 - 3\omega + 2)$$

$$+ \frac{3}{8}(2\omega^3 - 3\omega^2 + 1)$$

$$= \frac{1}{8}(\omega-1)^2[\omega^2(\omega^4+2\omega^3+3\omega^2-3)\cosh(4\Omega)+4\omega^2(\omega+2)\cosh(2\Omega)$$
$$+3(2\omega+1)],$$

where

$$\sinh^2(\Omega)=\frac{1}{2}[\cosh(2\Omega)-1],$$
$$\sinh^4(\Omega)=\frac{1}{8}[\cosh(4\Omega)-4\cosh(2\Omega)+3],$$
$$\sinh(3\Omega)\sinh(\Omega)=\frac{1}{2}[\cosh(4\Omega)-\cosh(2\Omega)],$$
$$\sinh^2(\Omega)\cosh(2\Omega)=\frac{1}{4}[\cosh(4\Omega)-2\cosh(2\Omega)+1],$$

and

$$\mathrm{E}(Y^3)=\mathrm{E}\left\{\left[\frac{1}{2}\exp\left(\frac{Z-c}{d}\right)-\frac{1}{2}\exp\left(-\frac{Z-c}{d}\right)\right]^4\right\}$$

$$= \frac{1}{16}\exp\left(\frac{-4c}{d}\right)\mathrm{E}\left[\exp\left(\frac{4Z}{d}\right)\right]-\frac{4}{16}\exp\left(\frac{-2c}{d}\right)\mathrm{E}\left[\exp\left(\frac{2Z}{d}\right)\right]$$
$$-\frac{4}{16}\exp\left(\frac{2c}{d}\right)\mathrm{E}\left[\exp\left(\frac{-2Z}{d}\right)\right]+\frac{1}{16}\exp\left(\frac{4c}{d}\right)\mathrm{E}\left[\exp\left(\frac{-4Z}{d}\right)\right]+\frac{6}{16}$$
$$=\frac{1}{16}e^{-4\Omega}\omega^8-\frac{1}{4}e^{-2\Omega}\omega^2-\frac{1}{4}e^{2\Omega}\omega^2+\frac{1}{16}e^{4\Omega}\omega^8+\frac{3}{8}$$
$$=\frac{1}{8}\omega^8\cosh(4\Omega)-\frac{1}{2}\omega^2\cosh(2\Omega)+\frac{3}{8},$$

$\Omega=c/d$, and $\omega=\exp(1/d^2)$.

Appendix C. Derivation of Theorem 1

If we adopt the US distribution to approximate the *GB* distribution, the pricing formula of the *GB* call option can be derived as follows:[12]

$$
\begin{aligned}
GBC(0) &= e^{-rT}\int_K^{\infty}(x-K)f_{\mathrm{US}}(x)dx \\
&= e^{-rT}\left[\int_K^{\infty}xf_{\mathrm{US}}(x)dx - K\int_K^{\infty}f_{\mathrm{US}}(x)dx\right] \\
&= e^{-rT}\left[\int_{-\infty}^{\infty}xf_{\mathrm{US}}(x)dx - \int_{-\infty}^{K}xf_{\mathrm{US}}(x)dx - K\int_{-\infty}^{\infty}f_{\mathrm{US}}(x)dx + K\int_{-\infty}^{K}f_{\mathrm{US}}(x)dx\right] \\
&= e^{-rT}\left[\mathcal{M} - K - \int_{-\infty}^{K}xf_{\mathrm{US}}(x)dx + K\int_{-\infty}^{K}f_{\mathrm{US}}(x)dx\right],
\end{aligned}
\tag{29}
$$

where M is given in (4) and $f_{\mathrm{US}}(x)$ is the probability density function of the US distribution presented in equation (10).

Based on the changing-variable technique and equation (8), the second integration in equation (29) can be straightforward derived as follows:

$$
\int_{-\infty}^{K}f_{\mathrm{US}}(x)dx = \int_{-\infty}^{R}\phi(z)dz = \Phi(R),
\tag{30}
$$

where $R = c + d\sinh^{-1}\left(\frac{K-a}{b}\right)$, $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{\frac{-1}{2}x^2}$, and $\Phi(y) = \int_{-\infty}^{y}\phi(z)dz$.

Similarly, the first integration in equation (29) can be derived as follows:

$$
\begin{aligned}
\int_{-\infty}^{K}xf_{\mathrm{US}}(x)dx &= \int_{-\infty}^{R}a + b\sinh\left(\frac{z-c}{d}\right)\phi(z)dz \\
&= a\int_{-\infty}^{R}a\phi(z)dz + \frac{b}{2}\int_{-\infty}^{R}\left[\exp\left(\frac{z-c}{d}\right) - \exp\left(\frac{c-z}{d}\right)\right]\phi(z)dz \\
&= a\Phi(R) + \frac{b}{2}\exp\left(\frac{1-2cd}{2d^2}\right)\Phi\left(R-\frac{1}{d}\right) - \frac{b}{2}\exp\left(\frac{1+2cd}{2d^2}\right)\Phi\left(R+\frac{1}{d}\right).
\end{aligned}
\tag{31}
$$

12 The approximation can be viewed as an application of the Edgeworth series expansion (see Cramér, 1946; Kendall and Stuart, 1977), which shows that a given probability distribution can be approximated by an arbitrary distribution in terms of a series expansion involving adjustments of second and higher moments. Jarrow and Rudd (1982) first employ the Edgeworth series expansion to price options with the lognormal as the approximating distribution. However, this article adopts the US distribution as the approximating distribution.

With equations (29), (30), and (31), the pricing formula of the *GB* call option can be obtained. The derivation of the pricing formula for the *GB* put option is similar to the call option and thus it has been omitted.

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